Problem Set 9 Solutions

Section 6.5

2) $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$ $\quad b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

$$A^tA = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$$

$$A^tb = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 12 & 8 & -24 \\ 8 & 10 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{2}{3} & -2 \\ 0 & \frac{14}{3} & 14 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 3 \end{array}\right] \quad \boxed{\hat{x} = (-4, 3)}$$

4) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

$$A^tA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^tb = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 11 & 14 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 8 & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\boxed{\hat{x} = (1,1)}$$

6) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = A \quad b = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$

$$A^tA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 6 & 3 & 3 & 27 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 1 & 1 & 0 & 4 \\ 1 & 0 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\hat{x} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$$

8) $\| (5,1,0) - A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \|$

$= \| (5,1,0) - (4,0,2) \| = \sqrt{1^2+1^2+2^2} = \boxed{\sqrt{6}}$

10) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

Note that $(1,-1,1), (2,4,2)$ is an orthogonal basis for Col(A).

$c_1 = \frac{\langle b, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{9}{3} = 3 \quad c_2 = \frac{\langle b, v_2 \rangle}{\langle v_2, v_2 \rangle} = \frac{12}{24} = \frac{1}{2}$

$\text{proj}_{Col(A)} b = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

$= \boxed{\begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}}$

$A\hat{x} = \text{proj}_{Col(A)} b$

$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ -1 & 4 & -1 \\ 1 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$

$\boxed{\hat{x} = (3, \tfrac{1}{2}, 0)}$

Section 7.1

10) $$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$ Columns form an ONB, so this matrix is orthogonal

$$A^{-1} = A^t = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = A$$

12) This matrix is orthogonal since the columns form an ONB.

$$A^{-1} = A^t = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix}$$

14) $$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} \qquad \begin{vmatrix} 1-x & -5 \\ -5 & 1-x \end{vmatrix} = x^2 - 2x - 24$$
$$= (x-6)(x+4)$$

$$\lambda = 6 \qquad \begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda = -4 \qquad \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

18) $\begin{bmatrix} 1 & -6 & 4 \\ -6 & 2 & -2 \\ 4 & -2 & -3 \end{bmatrix}$ $\lambda = -3, -6, 9$

$\lambda = -3$ $\begin{bmatrix} 4 & -6 & 4 \\ -6 & 5 & -2 \\ 4 & -2 & 0 \end{bmatrix}$ $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\lambda = -6$ $\begin{bmatrix} 7 & -6 & 4 \\ -6 & 8 & -2 \\ 4 & -2 & 3 \end{bmatrix}$ $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

$\lambda = 9$ $\begin{bmatrix} -8 & -6 & 4 \\ -6 & -7 & -2 \\ 4 & -2 & -12 \end{bmatrix}$ $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

These are already orthogonal

$B = \left\{ \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

20) $\begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$ $\lambda = -3, 15$

$\lambda = -3$ $\begin{bmatrix} 8 & 8 & -4 \\ 8 & 8 & -4 \\ -4 & -4 & 2 \end{bmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda = 15$ $\begin{bmatrix} -10 & 8 & -4 \\ 8 & -10 & -4 \\ -4 & -4 & -16 \end{bmatrix}$ $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$w_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$v_2 - \langle v_2, w_1 \rangle w_1 = (1,-1,0) - \frac{1}{5}(1)(1,0,2)$

$= \left(\frac{4}{5}, -1, -\frac{2}{5}\right) \Rightarrow (4,-5,-2)$

$w_2 = \frac{1}{3\sqrt{5}}(4,-5,-2)$

$w_3 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

$$A = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{2}{3} \\ 0 & -\frac{5}{3\sqrt{5}} & \frac{2}{3} \\ \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{4}{3\sqrt{5}} & -\frac{5}{3\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

27) Suppose A is a symmetric $n \times n$ matrix.

$\langle Ae_i, e_j \rangle$ is the (j,i) entry of A where e_i denotes the standard basis.

Since A is symmetric, the (i,j) entry and (j,i) entry of A are equal, so

$$\langle Ae_i, e_j \rangle = \langle Ae_j, e_i \rangle. \quad (*)$$

To show $\langle Av, w \rangle = \langle Aw, v \rangle$ in general, write

$$v = a_1 e_1 + \cdots + a_n e_n$$
$$w = b_1 e_1 + \cdots + b_n e_n$$

$$\begin{aligned}
\langle Av, w \rangle &= \langle A(a_1 e_1 + \cdots + a_n e_n), b_1 e_1 + \cdots + b_n e_n \rangle \\
&= \langle a_1 A e_1 + \cdots + a_n A e_n, b_1 e_1 + \cdots + b_n e_n \rangle \\
&= \sum_{i,j=1}^{n} a_i b_j \langle Ae_i, e_j \rangle \\
&\underset{\text{by } (*)}{=} \sum_{i,j=1}^{n} a_i b_j \langle Ae_j, e_i \rangle \\
&= \sum_{i,j=1}^{n} a_i b_j \langle e_i, Ae_j \rangle \\
&= \langle a_1 e_1 + \cdots + a_n e_n, b_1 A e_1 + \cdots + b_n A e_n \rangle \\
&= \langle a_1 e_1 + \cdots + a_n e_n, A(b_1 e_1 + \cdots + b_n e_n) \rangle \\
&= \langle v, Aw \rangle.
\end{aligned}$$

Additional Problems

1) Let A be a symmetric $n \times n$ matrix.
Then there is an orthonormal basis
$B = \{v_1, v_2, \ldots, v_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors
of A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

We can assume $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

Given any $v \neq 0$, we can write
$$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$
for c_i not all zero.

$$\langle v, v \rangle = c_1^2 + c_2^2 + \ldots + c_n^2$$
$$Av = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n$$
$$\langle Av, v \rangle = c_1^2 \lambda_1 + c_2^2 \lambda_2 + \ldots + c_n^2 \lambda_n$$

Then, since $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$,

$$\lambda_1 (c_1^2 + c_2^2 + \ldots + c_n^2) \leq c_1^2 \lambda_1 + c_2^2 \lambda_2 + \ldots + c_n^2 \lambda_n$$
$$\leq \lambda_n (c_1^2 + c_2^2 + \ldots + c_n^2)$$

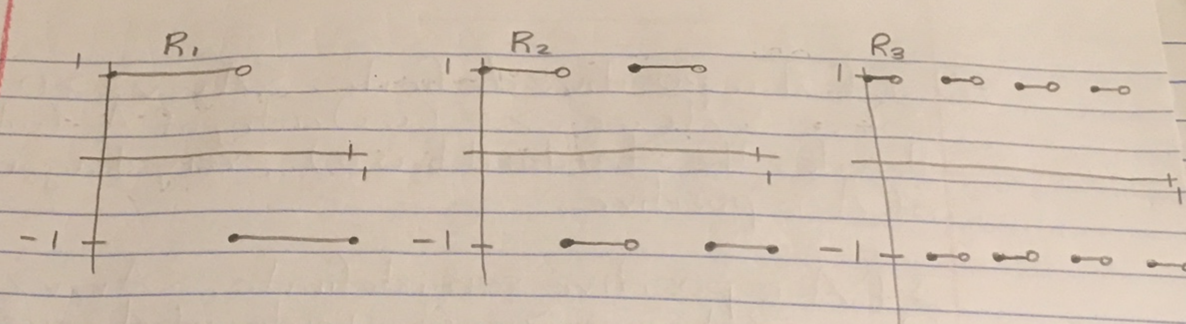
So $$\lambda_{min} = \lambda_1 \leq \frac{\langle Av, v \rangle}{\langle v, v \rangle} \leq \lambda_n = \lambda_{max}$$

for all $v \neq 0$.

For any eigenvector w_1 for λ_{min},
$$\frac{\langle Aw_1, w_1 \rangle}{\langle w_1, w_1 \rangle} = \frac{\langle \lambda_{min} w_1, w_1 \rangle}{\langle w_1, w_1 \rangle} = \lambda_{min}$$
and for any eigenvector w_2 for λ_{max},
$$\frac{\langle Aw_2, w_2 \rangle}{\langle w_2, w_2 \rangle} = \frac{\langle \lambda_{max} w_2, w_2 \rangle}{\langle w_2, w_2 \rangle} = \lambda_{max}$$

so indeed, $$\min_{v \neq 0, v \in \mathbb{R}^n} \frac{\langle Av, v \rangle}{\langle v, v \rangle} = \lambda_{min}, \quad \max_{v \neq 0, v \in \mathbb{R}^n} \frac{\langle Av, v \rangle}{\langle v, v \rangle} = \lambda_{max}.$$

2)



$R_1(x)\,R_1(x) = 1$ for $0 \le x \le 1$

$\langle R_1, R_1 \rangle = \int_0^1 1\,dx = 1$

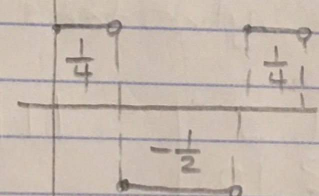
$R_2(x)\,R_2(x) = 1$ for $0 \le x \le 1$

$\langle R_2, R_2 \rangle = \int_0^1 1\,dx = 1$

$R_3(x)\,R_3(x) = 1$ for $0 \le x \le 1$

$\langle R_3, R_3 \rangle = \int_0^1 1\,dx = 1$

$R_1(x)\,R_2(x) =$

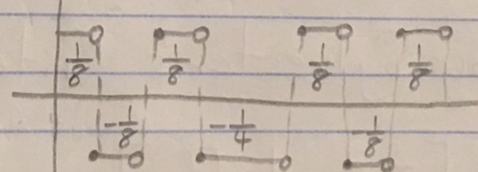


$\langle R_1, R_2 \rangle$

$= \frac{1}{4} + \left(-\frac{1}{2}\right) + \frac{1}{4}$

$= 0$

$R_1(x)\,R_3(x) =$

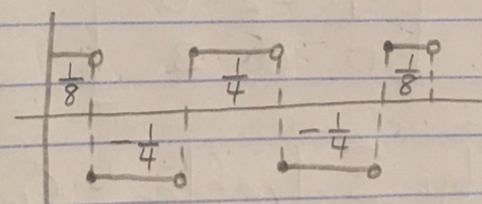


$\langle R_1, R_3 \rangle$

$= \frac{1}{8} - \frac{1}{8} + \frac{1}{8} - \frac{1}{4} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8}$

$= 0$

$R_2(x)\,R_3(x) =$



$\langle R_2, R_3 \rangle$

$= \frac{1}{8} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{8}$

$= 0$

So $\{R_1(x), R_2(x), R_3(x)\}$ is an orthonomal set in $L^2([0,1])$.

3) If A is positive definite, $\langle Av, v\rangle > 0$ for all $v \neq 0$. Since $\langle Ae_i, e_i\rangle = (i,i)$ entry of A (where e_i denotes the standard basis), every diagonal entry of A is positive.

If A is positive semidefinite, $\langle Av, v\rangle \geq 0$ for all $v \neq 0$. Since $\langle Ae_i, e_i\rangle = (i,i)$ entry of A, every diagonal entry of A is nonnegative.

4) $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ $\quad |1| = 1$ $\quad \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1$

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -1 & 4 \end{vmatrix} = 1(19) - 2(7) + (-1)(3) = 2$$

So by Sylvester's Criterion, A is positive definite.

$B = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}$ $\quad |1| = 1$ $\quad \begin{vmatrix} 1 & 3 \\ 3 & 10 \end{vmatrix} = 1$

So by Sylvester's Criterion, B is positive definite.

$C = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad |-2| = -2 \not> 0$

So by Sylvester's Criterion, C is not positive definite.

5) $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = A$ $\lambda = 0, -1$ so A is not positive semidefinite.

But $|0| = 0$

$\begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} = 0.$

6)

$$A = \begin{bmatrix} k & 1 & 0 & \cdots & 0 \\ 1 & k & 1 & \cdots & 0 \\ 0 & 1 & k & 1 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & & 1 \\ 0 & \cdots & 0 & 1 & k \end{bmatrix}$$

Let $v = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$Av = \begin{bmatrix} k+1 \\ k+1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad Aw = \begin{bmatrix} k-1 \\ 1-k \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So $\dfrac{\langle Av, v\rangle}{\langle v, v\rangle} = \dfrac{2(k+1)}{2} = k+1$

$\dfrac{\langle Aw, w\rangle}{\langle w, w\rangle} = \dfrac{2(k-1)}{2} = k-1$

So by Rayleigh's principle, $\lambda_{max} \geq k+1$ and $\lambda_{min} \leq k-1$.