Problem Set 8 Solutions
Section 6.1

13) $\|(10,-3),(-1,-5)\|$

$= \sqrt{11^2+2^2} = \sqrt{125} = \boxed{5\sqrt{5}}$

16) $u \cdot v = 24-9-15=0$ $\boxed{\text{orthogonal}}$

18) $u \cdot v = -3-56+60+0 = 1$ $\boxed{\text{not orthogonal}}$

24) $\langle u+v, u+v\rangle + \langle u-v, u-v\rangle$

$= \langle u, u+v\rangle + \langle v, u+v\rangle + \langle u, u-v\rangle - \langle v, u-v\rangle$

$= \langle u,u\rangle + \cancel{\langle u,v\rangle} + \cancel{\langle v,u\rangle} + \langle v,v\rangle$
$+ \langle u,u\rangle - \cancel{\langle u,v\rangle} - \cancel{\langle v,u\rangle} + \langle v,v\rangle$
$= 2\|u\|^2 + 2\|v\|^2$

29) $x \cdot v_j = 0$ for $1 \le j \le p$.

Check $x \cdot (c_1v_1 + c_2v_2 + \dots + c_pv_p) = 0$.

$x \cdot (c_1v_1 + c_2v_2 + \dots + c_pv_p)$

$= c_1 x\cdot v_1 + c_2 x \cdot v_2 + \dots + c_p x \cdot v_p$
$= c_1(0) + c_2(0) + \dots + c_p(0) = 0.$

31) If $x \in W$ and $x \in W^\perp$, x is orthogonal to every vector in W, but since $x \in W$, $x \cdot x = 0$. So by positive definiteness, $x = 0$.

Section 6.2

5) $(3,-2,1,3)\cdot(-1,3,-3,4)$
$= -3-6-3+12=0$
$(3,-2,1,3)\cdot(3,8,7,0) = 9-16+7+0=0$
$(-1,3,-3,4)\cdot(3,8,7,0) = -3+24-21+0=0$

$\boxed{\text{orthogonal set}}$

6) $(5,-4,0,3)\cdot(-4,1,-3,8)$
$= -20-4+0+24=0$
$(5,-4,0,3)\cdot(3,3,5,-1)$
$= 15-12+0-3=0$
$(-4,1,-3,8)\cdot(3,3,5,-1)$
$= -12+3-15-8=-32$

$\boxed{\text{not an orthogonal set}}$

10) $(3,-3,0)\cdot(2,2,-1) = 6-6+0=0$
$(3,-3,0)\cdot(1,1,4) = 3-3+0=0$
$(2,2,-1)\cdot(1,1,4) = 2+2-4=0$

$$c_1 = \frac{\langle x,u_1\rangle}{\langle u_1,u_1\rangle} = \frac{24}{18} = \frac{4}{3}$$

$$c_2 = \frac{\langle x,u_2\rangle}{\langle u_2,u_2\rangle} = \frac{3}{9} = \frac{1}{3} \quad c_3 = \frac{\langle x,u_3\rangle}{\langle u_3,u_3\rangle} = \frac{6}{18} = \frac{1}{3}$$

$$\boxed{x = \frac{4}{3}c_1 + \frac{1}{3}c_2 + \frac{1}{3}c_3}$$

12) $c = \frac{(1,-1)\cdot(-1,3)}{(-1,3)\cdot(-1,3)} = -\frac{4}{10} = -\frac{2}{5}$

$-\frac{2}{5}(-1,3) = \boxed{\left(\frac{2}{5}, -\frac{6}{5}\right)}$

14) $c = \frac{(2,6)\cdot(7,1)}{(7,1)\cdot(7,1)} = \frac{20}{50} = \frac{2}{5}$

$\frac{2}{5}(7,1) = \left(\frac{14}{5}, \frac{2}{5}\right)$

$(2,6) - \left(\frac{14}{5}, \frac{2}{5}\right) = \left(-\frac{4}{5}, \frac{28}{5}\right)$

$\boxed{(2,6) = \left(\frac{14}{5}, \frac{2}{5}\right) + \left(-\frac{4}{5}, \frac{28}{5}\right)}$

16) $\frac{(-3,9)\cdot(1,2)}{(1,2)\cdot(1,2)} = \frac{15}{5} = 3$

$\|(-3,9)-(3,6)\| = \sqrt{36+9} = \boxed{3\sqrt{5}}$

26) Any set of orthogonal vectors is linearly independent so W has n linearly independent vectors in $\mathbb{R}^n$ and since $\dim(\mathbb{R}^n) = n$, W forms a basis for $\mathbb{R}^n$.

Section 6.3

6) $u_1 \cdot u_2 = 0 - 1 + 1 = 0$

$c_1 = \frac{\langle y, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{-27}{18} = -\frac{3}{2}$ $c_2 = \frac{\langle y, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{5}{2}$

$-\frac{3}{2}u_1 + \frac{5}{2}u_2 = \boxed{(6,4,1)}$

10) Note that u_1, u_2, u_3 are orthogonal.

$c_1 = \frac{\langle y, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{1}{3}$ $\quad c_2 = \frac{\langle y, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{14}{3}$

$c_3 = \frac{\langle y, u_3 \rangle}{\langle u_3, u_3 \rangle} = -\frac{5}{3}$

$c_1 u_1 + c_2 u_2 + c_3 u_3 = (5, 2, 3, 6)$

$$\boxed{(3,4,5,6) = (5,2,3,6) + (-2,2,2,0)}$$

12) $v_1 \cdot v_2 = 0$ $\quad c_1 = \frac{\langle y, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{30}{10} = 3$

$c_2 = \frac{\langle y, v_2 \rangle}{\langle v_2, v_2 \rangle} = \frac{26}{26} = 1$

$3v_1 + v_2 = \boxed{(-1,-5,-3,9)}$

13) $v_1 \cdot v_2 = 0$

$c_1 = \frac{\langle z, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{10}{15} = \frac{2}{3}$ $\quad c_2 = \frac{\langle z, v_2 \rangle}{\langle v_2, v_2 \rangle} = \frac{-7}{3}$

$$\boxed{\frac{2}{3} v_1 - \frac{7}{3} v_2}$$

15) $u_1 \cdot u_2 = 0$

$c_1 = \frac{\langle y, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{35}{35} = 1$ $\quad c_2 = \frac{\langle y, u_2 \rangle}{\langle u_2, u_2 \rangle} = \frac{-28}{14} = -2$

$v_1 - 2v_2 = (3, -9, -1)$

$\| (5,-9,5) - (3,-9,-1) \|$
$= \| (2,0,6) \| = \sqrt{40} = \boxed{2\sqrt{10}}$

Section 6.4

7) $v_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ $\boxed{w_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}}$

$v_2 - \langle v_2, w_1 \rangle w_1 = (4,-1,2) - \frac{1}{\sqrt{30}} \cdot 15 \cdot \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$

$= (4,-1,2) - (1, -\frac{5}{2}, \frac{1}{2})$

$= (3, \frac{3}{2}, \frac{3}{2})$

$\Rightarrow \boxed{w_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}$

8) $v_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$ $\boxed{w_1 = \frac{1}{5\sqrt{2}} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}}$

$v_2 - \langle v_2, w_1 \rangle w_1 = (-3, 14, -7) - \frac{1}{5\sqrt{2}} \cdot (-100) \frac{1}{5\sqrt{2}} (3,-4,5)$

$= (3, 6, 3)$

$\boxed{w_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$

10) $\begin{bmatrix} -1 & 3 & 1 & 1 \\ 6 & -8 & -2 & -4 \\ 6 & 3 & 6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -1 & -1 \\ 0 & 10 & 4 & 2 \\ 0 & 11 & 8 & 7 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & -3 & -1 & -1 \\ 0 & 10 & 4 & 2 \\ 0 & 0 & 36 & 48 \end{bmatrix}$ $10R_3 - 11R_2$

$\rightarrow \begin{bmatrix} 1 & -3 & -1 & -1 \\ 0 & 10 & 4 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$

$v_1 = (1, -3, -1, -1) \Rightarrow \boxed{w_1 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \\ -3 \\ -1 \\ -1 \end{bmatrix}}$

$v_2 - \langle v_2, w_1 \rangle w_1 = (0, 10, 4, 2) - \frac{1}{2\sqrt{3}}(-36)\frac{1}{2\sqrt{3}}(1,-3,-1,-1)$ 3

$= (3, 1, 1, -1)$

$\boxed{w_2 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}}$

$v_3 - \langle v_3, w_1 \rangle w_1 - \langle v_3, w_2 \rangle w_2$

$= (0,0,3,4) - \frac{1}{12}(-7)(1,-3,-1,-1) - \frac{1}{12}(-1)(3,1,1,-1)$

$= \left(\frac{5}{6}, -\frac{5}{3}, \frac{15}{6}, \frac{10}{3}\right) \Rightarrow (5, -10, 15, 20)$

$\Rightarrow (1, -2, 3, 4)$

$\boxed{w_3 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}}$

12) $\begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 3 & -3 & 2 & 5 & 5 \\ 5 & 1 & 3 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 6 & 3 & -3 & 3 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$V_1 = (1, 0, 0, 0, 1)$ $\boxed{w_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$

$V_2 - \langle v_2, w_1 \rangle w_1 = (0, 1, 0, -1, 0) - \frac{1}{2}(0) w_1$
$= (0, 1, 0, -1, 0)$

$$\boxed{w_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}$$

$v_3 - \langle v_3, w_1 \rangle w_1 - \langle v_3, w_2 \rangle w_2$
$= (0, 0, 1, 1, 1) - \frac{1}{2}(1)(1, 0, 0, 0, 1)$
$- \frac{1}{2}(-1)(0, 1, 0, -1, 0)$

$= (-\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}) \Rightarrow (-1, 1, 2, 1, 1)$

$$\boxed{w_3 = \frac{1}{2\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}}$$

Additional Problems

1) (a) $\frac{1}{2}(\langle v+w, v+w\rangle - \langle v,v\rangle - \langle w,w\rangle)$

$= \frac{1}{2}(\cancel{\langle v,v\rangle} + \langle v,w\rangle + \langle w,v\rangle + \cancel{\langle w,w\rangle}$
$- \cancel{\langle v,v\rangle} - \cancel{\langle w,w\rangle})$

$= \frac{1}{2}(2\langle v,w\rangle) = \langle v,w\rangle$

(b) If T preserves inner products, then it preserves norms since

$$\|Tv\| = (\langle Tv, Tv\rangle)^{1/2} = (\langle v,v\rangle)^{1/2} = \|v\|$$

If T preserves norms, then by the polarization identity ($\langle v,w\rangle = \frac{1}{2}(\|v+w\|^2 - \|v\|^2 - \|w\|^2)$)

$$\langle Tv, Tw\rangle = \frac{1}{2}(\|Tv+Tw\|^2 - \|Tv\|^2 - \|Tw\|^2)$$

$$= \frac{1}{2}(\|T(v+w)\|^2 - \|Tv\|^2 - \|Tw\|^2)$$

$$= \frac{1}{2}(\|v+w\|^2 - \|v\|^2 - \|w\|^2)$$

$= \langle v,w\rangle$ so T preserves inner products.

(c) Since T is bijective, T^{-1} exists, and is a bijective linear operator.

Since T is an isometry,

$$\langle Tu, Tv\rangle = \langle u, v\rangle \text{ for all } u, v \in V$$

Since for $w_1 = Tu$, $w_2 = Tv$, we have $u = T^{-1}w_1$, $v = T^{-1}w_2$, the above identity states

$$\langle w_1, w_2\rangle = \langle T^{-1}w_1, T^{-1}w_2\rangle$$

for all $w_1, w_2 \in V$ (since for every $w_1, w_2 \in V$, there exist u, v such that $Tu = w_1$, $Tv = w_2$ since T is onto.)

(d) $I: V \to V$ is a bijective linear operator and it preserves inner products since

$$\langle Iv, Iw\rangle = \langle v, w\rangle \text{ for all } v, w \in V$$

since $Iv = v$, $Iw = w$.

(e) If T_1 and T_2 are both bijective linear operators on V, so is $T_2 \circ T_1$.

In addition,

$$\langle (T_2 \circ T_1)v, (T_2 \circ T_1)w\rangle$$
$$= \langle T_2(T_1 v), T_2(T_1 w)\rangle \overset{T_2 \text{ is an isometry}}{=} \langle T_1 v, T_1 w\rangle$$
$$\overset{T_1 \text{ is an isometry}}{=} \langle v, w\rangle \quad v, w \in V.$$

2) Since the columns of A form an orthonormal basis,

$$\underbrace{\begin{bmatrix} \text{— col 1 —} \\ \vdots \\ \text{— col n —} \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} | & & | \\ \text{col 1} & \cdots & \text{col n} \\ | & & | \end{bmatrix}}_{A} = I$$

$$= I.$$

Since $A^TA = I$, $A^{-1} = A^T$. So $AA^T = I$ also.

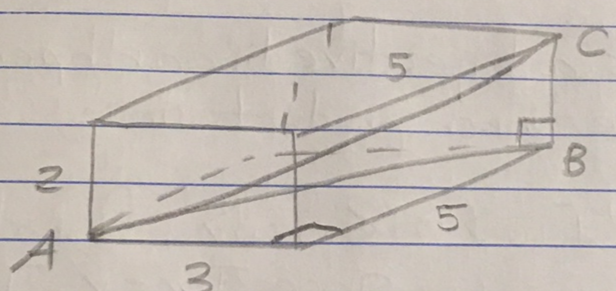
So

$$\underbrace{\begin{bmatrix} \text{— row 1 —} \\ \vdots \\ \text{— row n —} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | & & | \\ \text{row 1} & \text{—} & \text{row n} \\ | & & | \end{bmatrix}}_{A^T} = I$$

$$= I$$

So the rows of A form an orthonormal basis for $\mathbb{R}^n$.

3)



$BC = 2$

$AB = \sqrt{3^2 + 5^2}$

diagonal length

$= \sqrt{2^2 + (\sqrt{3^2+5^2})^2}$

$= \sqrt{2^2 + 3^2 + 5^2}$.

So similarly, $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

4) $\ell(1,0,0)=2 \quad \ell(1,1,1)=-1 \quad \ell(1,-1,1)=1$

Let $w=(x_1, x_2, x_3)$

$(1,0,0)\cdot(x_1,x_2,x_3)=2$
$(1,1,1)\cdot(x_1,x_2,x_3)=-1$
$(1,-1,1)\cdot(x_1,x_2,x_3)=1$

$$\begin{aligned} x_1 \qquad\quad &= 2 \\ x_1+x_2+x_3 &= -1 \Rightarrow \quad x_2+x_3=-3 \\ x_1-x_2+x_3 &= 1 \Rightarrow \quad x_2-x_3=1 \end{aligned}$$

$x_2=-1 \quad x_3=-2$

$$\boxed{w=(2,-1,-2)}$$

5) $B=\{v_1, v_2, \dots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$.
So $[T]_{B\to B}$

$$= \begin{bmatrix} \langle Tv_1, v_1\rangle & \langle Tv_2, v_1\rangle & \cdots & \langle Tv_n, v_1\rangle \\ \langle Tv_1, v_2\rangle & \langle Tv_2, v_2\rangle & \cdots & \langle Tv_n, v_2\rangle \\ \vdots & & & \\ \langle Tv_1, v_n\rangle & \langle Tv_2, v_n\rangle & \cdots & \langle Tv_n, v_n\rangle \end{bmatrix}$$

(a) If T is a symmetric linear operator,
$\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in \mathbb{R}^n$.

So $\underbrace{\langle Tv_i, v_j\rangle}_{(j,i)\text{ entry of } [T]_{B\to B}} = \langle v_i, Tv_j\rangle = \underbrace{\langle Tv_j, v_i\rangle}_{(i,j)\text{ entry of } [T]_{B\to B}}$.

So $[T]_{B\to B}$ is a symmetric matrix.

(b) Since $[T]_{B\to B}$ is a symmetric matrix,

$$(*) \quad \langle Tv_i, v_j\rangle = \langle Tv_j, v_i\rangle \text{ for all } 1 \le i,j \le n.$$

We need to show in general that $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in \mathbb{R}^n$.

Since $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$,

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$$
$$w = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$$

$$\langle Tv, w\rangle = \langle T(a_1 v_1 + \ldots + a_n v_n), b_1 v_1 + \ldots + b_n v_n\rangle$$

$$= \langle a_1 T v_1 + \ldots + a_n T v_n, b_1 v_1 + \ldots + b_n v_n\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \langle T v_i, v_j\rangle$$

$$\overset{(*)}{=} \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \langle v_i, T v_j\rangle$$

$$= \langle a_1 v_1 + a_2 v_2 + \ldots + a_n v_n, b_1 T v_1 + b_2 T v_2 + \ldots + b_n T v_n\rangle$$

$$= \langle a_1 v_1 + \ldots + a_n v_n, T(b_1 v_1 + \ldots + b_n v_n)\rangle$$

$$= \langle v, Tw\rangle.$$

7) $v_1 = 1$ $\quad \langle 1, 1\rangle = \int_0^1 1 \cdot 1\,dx = 1$ $\quad \boxed{w_1 = 1}$

$v_2 = x$ $\quad \langle v_2, w_1\rangle = \int_0^1 x \cdot 1\,dx = \frac{1}{2}$

$v_2 - \langle v_2, w_1\rangle w_1 = x - \frac{1}{2}$

$\int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{3}(x - \frac{1}{2})^3 \Big|_0^1 = \frac{1}{24} + \frac{1}{24} = \frac{1}{12}$

$$\boxed{w_2 = 2\sqrt{3}\left(x - \frac{1}{2}\right)}$$

$v_3 = x^2$ $\quad \langle v_3, w_1\rangle = \int_0^1 x^2 dx = \frac{1}{3}$

$\langle v_3, w_2\rangle = 2\sqrt{3}\int_0^1 x^2(x - \frac{1}{2})dx$

$= 2\sqrt{3}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{\sqrt{3}}{6}$

$x^2 - \frac{1}{3}(1) - \frac{\sqrt{3}}{6} 2\sqrt{3}\left(x - \frac{1}{2}\right)$

$= x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}$

$\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx$

$= \int_0^1 x^4 - x^3 + \frac{1}{6}x^2 - x^3 + x^2 - \frac{1}{6}x + \frac{1}{6}x^2 - \frac{1}{6}x + \frac{1}{36}\,dx$

$= \int_0^1 x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\,dx$

$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} = \frac{36 - 90 + 80 - 30 + 5}{180}$

$= \frac{1}{180}$

$$\boxed{w_3 = 3\sqrt{20}\left(x^2 - x + \frac{1}{6}\right)}$$