Problem Set 8

Section 5.1

13) $A - I = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \Rightarrow \lambda = 1 \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$A - 2I = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \Rightarrow \lambda = 2 \quad \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

$A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \Rightarrow \lambda = 3 \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

16) $A - 4I = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

18) $\lambda = 0, -3, 4$ (diagonal entries are eigenvalues since matrix is lower triangular)

19) $\lambda = 6$ since $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$.

25) If $Av = \lambda v$, then $A^{-1}(\lambda v) = v$.
$\lambda \neq 0$ since A is invertible and hence has trivial nullspace. So

$$\lambda A^{-1} v = v \Rightarrow A^{-1} v = \lambda^{-1} v.$$

26) If λ is an eigenvalue of A with eigenvector v,

$$A^2 v = \lambda^2 v = 0 \text{ so } \lambda^2 = 0 \Rightarrow \lambda = 0.$$

Section 5.2

14) $\text{char}_A(x)$

$$= \begin{vmatrix} 5-x & -2 & 3 \\ 0 & 1-x & 0 \\ 6 & 7 & -2-x \end{vmatrix}$$

$$= (1-x)((5-x)(-2-x)-18)$$
$$= (1-x)(x^2-3x-28)$$

16) lower triangular, so $\lambda = 5, -4, 1, 1$

18) $A - 5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & 0 & h-6 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 0 & h-6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

need two free variables so $\boxed{h=6}$

20) $\text{char}_A(x) = \det(A - xI)$

$$= \det(A - xI)^t$$
$$= \det(A^t - xI^t)$$
$$= \det(A^t - xI) = \text{char}_{A^t}(x)$$

Section 5.3

4) $A^k = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$

6) $\lambda = 5 \quad \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 4 \quad \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

15) $\begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix} = A$

$\lambda = 3 \quad A - 3I = \begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Basis: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1 \quad A - I = \begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix}$ Basis: $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

$$\begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

20) $\lambda = 4$ $\quad A - 4I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$ $\quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 2$ $\quad A - 2I = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}^{-1}$$

27) $A = PDP^{-1}$ so $A^{-1} = PD^{-1}P^{-1}$

where D^{-1} exists since none of the eigenvalues of A is zero since A is invertible.

32) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \lambda = 0, 1$

Additional Problems

1) $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{vmatrix} 1-x & 1 \\ 1 & -x \end{vmatrix} \qquad \mathrm{char}_M(x) = (-x + x^2 - 1)$$
$$= x^2 - x - 1$$

$$\frac{1 \pm \sqrt{5}}{2} = \lambda$$

$$\lambda = \frac{1+\sqrt{5}}{2} \Rightarrow M - \lambda I = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$\begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}$$

$$\lambda = \frac{1-\sqrt{5}}{2} \Rightarrow M - \lambda I = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{\sqrt{5}-1}{2} \end{bmatrix}$$

$$\begin{pmatrix} -1 \\ \frac{\sqrt{5}+1}{2} \end{pmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ \frac{\sqrt{5}-1}{2} & \frac{\sqrt{5}+1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\sqrt{5}+1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{\sqrt{5}-1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{\sqrt{5}+1}{2} & 1 \\ -\frac{\sqrt{5}-1}{2} & 1 \end{bmatrix}$$

$$M = \begin{pmatrix} 1 & -1 \\ \frac{\sqrt{5}-1}{2} & \frac{\sqrt{5}+1}{2} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{\sqrt{5}+1}{2} & 1 \\ -\frac{\sqrt{5}-1}{2} & 1 \end{pmatrix}$$

$$\begin{bmatrix} a_{1001} \\ a_{1000} \end{bmatrix} = M^{1000} \underset{\substack{\uparrow \\ \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$= \begin{pmatrix} 1 & -1 \\ \frac{\sqrt{5}-1}{2} & \frac{\sqrt{5}+1}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{1000} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{1000} \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{\sqrt{5}+1}{2} & 1 \\ -\frac{\sqrt{5}-1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ \frac{\sqrt{5}-1}{2} & \frac{\sqrt{5}+1}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{1000} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{1000} \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{\sqrt{5}+3}{2} \\ \frac{-\sqrt{5}+3}{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -1 \\ \frac{\sqrt{5}-1}{2} & \frac{\sqrt{5}+1}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{1000} \frac{\sqrt{5}+3}{2} \\ \left(\frac{1-\sqrt{5}}{2}\right)^{1000} \left(\frac{3-\sqrt{5}}{2}\right) \end{pmatrix}$$

$$\boxed{a_{1000} = \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}-1}{2} \left(\frac{1+\sqrt{5}}{2}\right)^{1000} \frac{\sqrt{5}+3}{2} + \frac{\sqrt{5}+1}{2} \left(\frac{1-\sqrt{5}}{2}\right)^{1000} \frac{3-\sqrt{5}}{2} \right)}$$

2) $T(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots)$

$(a_1, a_2, a_3, \ldots) = \lambda(a_0, a_1, a_2, \ldots)$

$a_1 = \lambda a_0$
$a_2 = \lambda a_1$
$a_3 = \lambda a_2$
$\vdots$

$a_0, a_1, a_2, a_3, \ldots$
$\times\lambda \quad \times\lambda \quad \times\lambda$

So every nonzero geometric sequence is an eigenvector. Since for any real λ, $1, \lambda, \lambda^2, \lambda^3, \ldots$ is a nonzero geometric sequence with factor λ, every real number is an eigenvalue.

3) (a) If λ is an eigenvalue of T, then $\ker(T-\lambda I)$ is nontrivial so $T-\lambda I$ is not invertible.

(b) If $T-\lambda I$ is not invertible, we have two cases.

① If $T-\lambda I$ is not one-to-one, $\ker(T-\lambda I)$ is nontrivial, and hence λ is an eigenvalue of T.

② If $T-\lambda I$ is not onto, then $\text{rank}(T-\lambda I) < n$ where $n = \dim(V)$ is finite. Then by Rank-Nullity,

$$\text{nullity}(T-\lambda I) + \underbrace{\text{rank}(T-\lambda I)}_{<n} = \overset{\dim V}{n}$$

so $\text{nullity}(T-\lambda I) \geq 1$. So $T-\lambda I$ has nontrivial kernel and hence λ is an eigenvalue of T.

2) $T(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots)$

$(a_1, a_2, a_3, \dots) = \lambda(a_0, a_1, a_2, \dots)$

$a_1 = \lambda a_0$

$a_2 = \lambda a_1$

$a_3 = \lambda a_2$

$\vdots$

$$\underbrace{a_0, a_1}_{\times\lambda}, \underbrace{a_1, a_2}_{\times\lambda}, \underbrace{a_2, a_3}_{\times\lambda}, \dots$$

So every nonzero geometric sequence is an eigenvector. Since for any real λ, $1, \lambda, \lambda^2, \lambda^3, \dots$ is a nonzero geometric sequence with factor λ, every real number is an eigenvalue.

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(c) Same as (a).

(d) Consider $T: V \to V$ given by

$$T(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$$

where V is the vector space of infinite sequences. Note that T is a linear operator on an infinite dimensional vector space.

For $\lambda = 0$, $T - \lambda I = T$ is not invertible, since T is not onto (range is only sequences in V with first coordinate equal to 0).

But $\lambda = 0$ is not an eigenvalue, since

$$\ker(T - \lambda I) = \ker(T) = \{\text{zero sequence}\}$$

so there is no nonzero sequence v such that

$$Tv = 0v = 0$$

So $\lambda = 0$ is not an eigenvalue.

4) (a) Use $B = \{1, x, x^2, x^3\}$ $\frac{d}{dx}(x \cdot 1) = 1$ $\frac{d}{dx}(x \cdot x) = 2x$
$\frac{d}{dx}(x \cdot x^2) = 3x^2$ $\frac{d}{dx}(x \cdot x^3) = 4x^3$

$$[T]_{B \to B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(b) $\det(T) = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ $\quad tr(T) = 1 + 2 + 3 + 4 = 10$

$$\begin{vmatrix} 1-x & 0 & 0 & 0 \\ 0 & 2-x & 0 & 0 \\ 0 & 0 & 3-x & 0 \\ 0 & 0 & 0 & 4-x \end{vmatrix} = (1-x)(2-x)(3-x)(4-x) = char_A(x)$$

$\lambda = 1$ $t\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \Rightarrow p(x) = t \cdot 1$

$\lambda = 2$ $t\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \Rightarrow p(x) = tx$

$\lambda = 3$ $t\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \Rightarrow p(x) = tx^2$

$\lambda = 4$ $t\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \Rightarrow p(x) = tx^3$