Problem Set 6 Solutions

Section 4.4   4, 8, 14, 19

4)
$$-4\begin{bmatrix}-1\\2\\0\end{bmatrix}+8\begin{bmatrix}3\\-5\\2\end{bmatrix}+(-7)\begin{bmatrix}4\\-7\\3\end{bmatrix}$$
$$=\begin{bmatrix}0\\1\\-5\end{bmatrix}$$

8)
$$c_1\begin{bmatrix}1\\0\\3\end{bmatrix}+c_2\begin{bmatrix}2\\1\\8\end{bmatrix}+c_3\begin{bmatrix}1\\-1\\2\end{bmatrix}=\begin{bmatrix}3\\-5\\4\end{bmatrix}$$

$$c_1+2c_2+c_3=3$$
$$c_2-c_3=-5$$
$$3c_1+8c_2+2c_3=4$$

$$\left[\begin{array}{ccc|c}1&2&1&3\\0&1&-1&-5\\3&8&2&4\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&2&1&3\\0&1&-1&-5\\0&2&-1&-5\end{array}\right]R_3-3R_1$$

$$\rightarrow\left[\begin{array}{ccc|c}1&0&2&8\\0&1&-1&-5\\0&0&1&5\end{array}\right]\begin{array}{l}R_1-R_3\\ \\ R_3-2R_2\end{array}\rightarrow\left[\begin{array}{ccc|c}1&0&0&-2\\0&1&0&0\\0&0&1&5\end{array}\right]\begin{array}{l}R_1-2R_3\\R_2+R_3\\ \end{array}$$

$$\begin{bmatrix}-2\\0\\5\end{bmatrix}B$$

14) $c_1(1-t^2)+c_2(t-t^2)+c_3(2-2t+t^2)$
$=(c_1+2c_3)+(c_2-2c_3)t+(-c_1-c_2+c_3)t^2$
$=3+t-6t^2$

$$\begin{aligned} c_1 \quad\quad + 2c_3 &= 3 \\ c_2 - 2c_3 &= 1 \\ -c_1 - c_2 + c_3 &= -6 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 3 & -3 \end{array}\right] R_1+R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] R_2+R_3 \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} R_1-2R_3 \\ R_2+2R_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}_B$$

19) The given property implies every element of V is a linearly combination of vectors in S so S spans V. 0 is a unique linear combination of vectors in S, so in particular

$$0 = 0v_1 + 0v_2 + \ldots + 0v_k$$

where $v_1, \ldots, v_k$ are the vectors in S and this is the uniquely linear combination for 0 in terms of vectors in S. So the vectors in S are linearly independent. So S is a basis for V.

Section 4.7 6, 8, 14

6) (a) $[I]_{F \to D} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

(b)

$$[x]_D = [I]_{F \to D} [x]_F$$

$$= \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}_D}$$

8) $(-1, 8) = c_1(1, 4) + c_2(1, 1)$

$c_1 + c_2 = -1 \qquad c_1 = 3 \quad c_2 = -4$

$4c_1 + c_2 = 8$

$(1, -5) = c_1(1, 4) + c_2(1, 1)$

$c_1 + c_2 = 1 \qquad c_1 = -2 \quad c_2 = 3$

$4c_1 + c_2 = -5$

$$\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = [I]_{B \to C}$$

$$[I]_{C \to B} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

14) $1 - 3t^2 = 1(1) + 0(t) + (-3)t^2$

$2 + t - 5t^2 = 2(1) + 1(t) + (-5)t^2$

$1 + 2t = 1(1) + 2(t) + (0)t^2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

$c_1(1 - 3t^2) + c_2(2 + t - 5t^2) + c_3(1 + 2t) = t^2$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{array}\right]$

$c_1 + 2c_2 + c_3 = 0$

$c_2 + 2c_3 = 0$

$-3c_1 - 5c_2 = 1$

$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right] R_3 + 3R_1$

$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\boxed{x^2 = 3(1 - 3t^2) + (-2)(2 + t - 5t^2) + 1(1 + 2t)}$

$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

Additional Problems

1) (a) If T is 1-1, $\dim(\ker(T)) = 0$

So $\underbrace{\text{nullity}(T)}_{0} + \text{rank}(T) = \dim(V)$

and hence $\dim(\text{range}(T)) = \dim(V)$
so since range (T) is a subspace of V,
range $(T) = V$ so T is onto.

(b) If T is onto, $\dim(\text{range}(T)) = \dim V$.
Hence, $\text{nullity}(T) + \text{rank}(T) = \dim V$
and so $\text{nullity}(T) = \dim(\ker(T)) = 0$.
So $\ker(T)$ is the zero subspace and
T is one-to-one.

(c) onto but not 1-1

$T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x_1, x_2, x_3) = (x_1, x_2)$

one-to-one but not onto

$T : \mathbb{R}^2 \to \mathbb{R}^3$ $T(x_1, x_2) = (x_1, x_2, 0)$

2) (a) Let $w_1, w_2 \in$ range (T). We need to show that $c_1 w_1 + c_2 w_2 \in$ range (T) too.

Since $w_1, w_2 \in$ range (T), for some
$v_1, v_2 \in V$, $T(v_1) = w_1$, $T(v_2) = w_2$.

By linearity, $T(c_1 v_1 + c_2 v_2) = c_1 w_1 + c_2 w_2$
so $c_1 w_1 + c_2 w_2 \in$ range (T).

(b) Define $T: M_{n\times n} \to \mathbb{R}^2$ by

$$T(M) = \begin{bmatrix} \text{sum of entries in first row} \\ \text{sum of entries in second col} \end{bmatrix}$$

range(T) has $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

since $T\left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

$T\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & \\ \vdots & \vdots & 0 & \\ 0 & 0 & & \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So range $(T) = \mathbb{R}^2$ since it is a subspace of $\mathbb{R}^2$ containing $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So rank $(T) = 2$. Thus, since $\dim(M_{n\times n}) = n^2$, nullity $(T) = \dim(\ker(T)) = n^2 - 2$.

Since $S = \ker(T)$, $\boxed{\dim S = n^2 - 2}$

(c) Define $T: P_{40} \to \mathbb{R}$ by

$T(p(x))$ = sum of coeff. on even powers of p.

T is onto since $ax^2 \to a$ by T for any $a \in \mathbb{R}$. So range $(T) = \mathbb{R}$, rank $(T) = 1$.
Since $\dim(P_{40}) = 41$, nullity $(T) = 40$.

Since $S = \ker(T)$, $\boxed{\dim(S) = 40.}$

3) (a) Vector addition: add sequences componentwise

The sum of two eventually zero sequences is also eventually zero.

$$\begin{array}{l} a_1, a_2, a_3, \ldots \quad 0, 0, 0, \ldots \\ + \; b_1, b_2, b_3, \ldots, 0, 0, 0, 0, 0, \ldots \\ \hline a_1+b_1, a_2+b_2, a_3+b_3, \ldots, 0, 0, \ldots 0, \ldots \end{array}$$

Scalar mult: multiply every element in sequence by the number.

$c(a_1, a_2, \ldots, 0, 0, 0, \ldots)$

$= ca_1, ca_2, \ldots, 0, 0, 0, \ldots$

is still eventually zero.

Zero vector is $0, 0, 0, \ldots, 0, 0, 0, \ldots$
which is also eventually zero.

(b) See the hint, which gives the proof outline.

4) Let $T: \mathbb{R}^n \to \mathbb{R}$ be a linear transformation.

Let $T((1,0,0,\_,0)) = a_1$

$T((0,1,0,\_,0)) = a_2$

$\vdots$

$T((0,0,0,\_,1)) = a_n$.

Then by linearity,

$T(x_1, x_2, \_, x_n)$

$= x_1 T(1,0,\_,0) + x_2 T(0,1,\_,0) + \ldots + x_n T(0,0,\_,1)$

$= a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$

$= (a_1, a_2, \_, a_n) \cdot (x_1, x_2, \_, x_n)$

$\uparrow$

dot product

where $a_1, a_2, \_, a_n$ can be any reals.

5) (a) Show span.

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$2c_1 + c_2 - c_3 = a$$
$$c_1 + 2c_2 = b$$
$$-c_2 + c_3 = c$$

But this always has a soln since

$$\begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = -(-1)(1) + 1(3) = 4 \neq 0.$$

Show lin. ind.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$2c_1 + c_2 - c_3 = 0$$
$$c_1 + 2c_2 = 0$$
$$-c_2 + c_3 = 0$$

But since $\det(A) \neq 0$, this only has the trivial solution.

$$[I]_{C_2 \to C_1} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[I]_{C_1 \to C_2} = [I]_{C_2 \to C_1}^{-1}$$

$$= \begin{bmatrix} 1/2 & 0 & 1/2 \\ -1/4 & 1/2 & -1/4 \\ -1/4 & 1/2 & 3/4 \end{bmatrix}$$

(b) $A - A^t$ is in $Skew_{3\times3}$ since

$$(A - A^t)^t = A^t - (A^t)^t = A^t - A$$
$$= -(A - A^t).$$

$T(A) = A - A^t$ is a linear transformation from $M_{3\times3}$ to $Skew_{3\times3}$.

$$T(c_1 A + c_2 B) = (c_1 A + c_2 B) - (c_1 A + c_2 B)^t$$

$$= c_1 (A - A^t) + c_2 (B - B^t)$$

$$= c_1 T(A) + c_2 T(B) \checkmark$$

$$T\left(\begin{bmatrix} 1&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}\right) = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}\right) = \begin{bmatrix} 0&1&0\\ -1&0&0\\ 0&0&0 \end{bmatrix} = \begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}_{C_1}$$

$$= [I]_{C_1 \to C_2} \begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}_{C_1}$$

$$= \begin{bmatrix} 1/2 & 0 & 1/2\\ -1/4 & 1/2 & -1/4\\ -1/4 & 1/2 & 3/4 \end{bmatrix} \begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix} = \begin{bmatrix} 1/2\\ -1/4\\ -1/4 \end{bmatrix}_{C_2}$$

$$T\left(\begin{bmatrix} 0&0&1\\ 0&0&0\\ 0&0&0 \end{bmatrix}\right) = \begin{bmatrix} 0&0&1\\ 0&0&0\\ -1&0&0 \end{bmatrix} = \begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}_{C_2}$$

$$= [I]_{C_1 \to C_2} \begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}_{C_2} = \begin{bmatrix} 0\\ 1/2\\ 1/2 \end{bmatrix}_{C_2}$$

$$T\left(\begin{bmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{bmatrix}\right) = \begin{bmatrix} 0&-1&0\\ 1&0&0\\ 0&0&0 \end{bmatrix} = \begin{bmatrix} -1\\ 0\\ 0 \end{bmatrix}_{C_1}$$

$$= [I]_{C_1 \to C_2} \begin{bmatrix} -1\\ 0\\ 0 \end{bmatrix}_{C_1} = \begin{bmatrix} -1/2\\ 1/4\\ 1/4 \end{bmatrix}_{C_2}$$

$$T\left(\begin{bmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{bmatrix}\right) = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0&0&0\\ 0&0&1\\ 0&0&0 \end{bmatrix}\right) = \begin{bmatrix} 0&0&0\\ 0&0&1\\ 0&-1&0 \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix}_{C_1} = [I]_{C_1 \to C_2} \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix}_{C_1} = \begin{bmatrix} 1/2\\ -1/4\\ 3/4 \end{bmatrix}_{C_2}$$

$$T\left(\begin{bmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{bmatrix}\right) = \begin{bmatrix} 0&0&-1\\ 0&0&0\\ 1&0&0 \end{bmatrix} = \begin{bmatrix} 0\\ -1\\ 0 \end{bmatrix}_{C_1} = [I]_{C_1 \to C_2} \begin{bmatrix} 0\\ -1\\ 0 \end{bmatrix}_{C_1} = \begin{bmatrix} 0\\ -1/2\\ -1/2 \end{bmatrix}_{C_2}$$

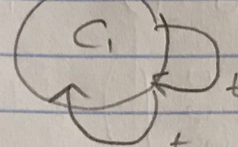
$$T\left(\begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&1&0 \end{bmatrix}\right) = \begin{bmatrix} 0&0&0\\ 0&0&-1\\ 0&1&0 \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ -1 \end{bmatrix}_{C_1} = [I]_{C_1 \to C_2} \begin{bmatrix} 0\\ 0\\ -1 \end{bmatrix}_{C_1} = \begin{bmatrix} -1/2\\ 1/4\\ -3/4 \end{bmatrix}_{C_2}$$

$$T\left(\begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{bmatrix}\right) = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}$$

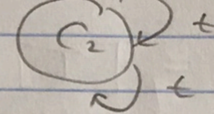
$$[T]_{B \to C_1} = \begin{bmatrix} 0&1&0&-1&0&0&0&0&0\\ 0&0&1&0&0&0&-1&0&0\\ 0&0&0&0&0&1&0&-1&0 \end{bmatrix} \quad [T]_{B \to C_2} = \begin{bmatrix} 0&1/2&0&-1/2&0&1/2&0&-1/2&0\\ 0&-1/4&1/2&1/4&0&-1/4&-1/2&1/4&0\\ 0&-1/4&1/2&1/4&0&3/4&-1/2&-3/4&0 \end{bmatrix}$$

(c) Makes sense since $t \circ t = \mathrm{id}_{\text{Skew } 3\times 3}$.

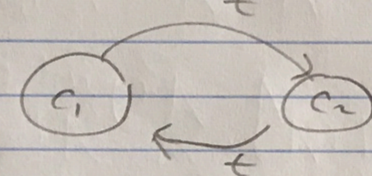
$[t]^2_{C_1 \to C_1}$ is



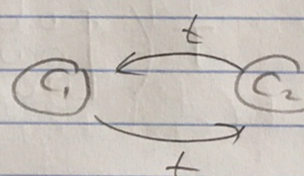
$[t]^2_{C_2 \to C_2}$ is



$[t]_{C_2 \to C_1}[t]_{C_1 \to C_2}$ is



and $[t]_{C_1 \to C_2}[t]_{C_2 \to C_1}$ is



$$[t]_{C_1 \to C_1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$[t]_{C_1 \to C_2} = [I]_{C_1 \to C_2}[t]_{C_1 \to C_1} = \begin{bmatrix} -1/2 & 0 & -1/2 \\ 1/4 & -1/2 & 1/4 \\ 1/4 & -1/2 & -3/4 \end{bmatrix}$$

$$[t]_{C_2 \to C_1} = [t]_{C_1 \to C_1}[I]_{C_2 \to C_1} = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$[t]_{C_2 \to C_2} = [I]_{C_1 \to C_2}[t]_{C_2 \to C_1}$$

$$= \begin{bmatrix} 1/2 & 0 & 1/2 \\ -1/4 & 1/2 & -1/4 \\ -1/4 & 1/2 & 3/4 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(d) This is because the coordinates do not match up correctly for $[t]^2_{C_1 \to C_2}$ for example. To be the identity matrix, $t \circ t$ needs to bring C_1 coordinates back to C_1 coordinates. But $[t]^2_{C_1 \to C_2}$ does not do this because it brings C_1 to C_2 coordinates and tries to do this again when it does not make sense logically (since we are in C_2 coordinates after the first map).