Problem Set 5 Solutions

Section 4.3   8, 11, 16, 23

8) not a basis since must be linearly dependent (more vectors than $\dim(\mathbb{R}^3)$)

$$\begin{bmatrix} 1 & -4 & 3 \\ 0 & 3 & -1 \\ 3 & -5 & 4 \\ 0 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 3 \\ 0 & 3 & -1 \\ 0 & 7 & -5 \\ 0 & 1 & -1 \end{bmatrix} \begin{array}{l} \\ \\ R_3-3R_1 \\ \frac{1}{2}R_4 \end{array} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{array}{l} R_1+4R_4 \\ R_2-3R_4 \\ R_3-7R_4 \\ \\ \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \boxed{\text{spans } \mathbb{R}^3}$$

11) $x + 2\underset{\substack{\uparrow \\ \text{free}}}{y} + \underset{\substack{\uparrow \\ \text{free}}}{z} = 0$

$$(-2s-t,\ s,\ t)$$

$$= s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

16) $$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -2 & 1 & -1 & 1 \\ 6 & -1 & 2 & -1 \\ 5 & -3 & 3 & -4 \\ 0 & 3 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & 2 & -7 \\ 0 & -3 & 3 & -9 \\ 0 & 3 & -1 & 1 \end{bmatrix} \begin{array}{l} \\ R_2+2R_1 \\ R_3-6R_1 \\ R_4-5R_1 \\ R_5 \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -8 \end{bmatrix} \begin{array}{l} \\ \\ R_3+R_2 \\ R_4+3R_2 \\ R_5-3R_2 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ R_2+R_3 \\ \\ \\ \\ \end{array} \qquad \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}}$$

23) Since $\dim(\mathbb{R}^4) = 4$, any 4 vectors that span $\mathbb{R}^4$ are necessarily linearly independent and hence form a basis for $\mathbb{R}^4$.

Section 4.5 9, 12, 16

9) $\{(a, b, a)\}$

$$(a,b,a) = a(1,0,1) + b(0,1,0)$$

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$ is a basis

(span shown above), can easily check they are linearly independent

$\boxed{\dim = 2}$

12)
$$\begin{bmatrix} 1 & -2 & 0 \\ -3 & 4 & 1 \\ -8 & 6 & 5 \\ -3 & 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \\ 0 & -10 & 5 \\ 0 & -6 & 7 \end{bmatrix} \begin{matrix} \\ R_2 + 3R_1 \\ R_3 + 8R_1 \\ R_4 + 3R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 3R_2 \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\dim = 3}$

16) $A = \begin{bmatrix} 3 & 4 \\ -6 & 10 \end{bmatrix} \to \begin{bmatrix} 3 & 4 \\ 0 & 18 \end{bmatrix} \to \begin{bmatrix} 3 & 0 \\ 0 & 18 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $\boxed{\dim(\mathrm{Nul}(A)) = 0}$

$(3, -6), (4, 10)$ span $\mathbb{R}^2$ so

$\boxed{\dim(\mathrm{Col}(A)) = 2}$

Section 4.6 2, 6, 13, 16

rank(A) = 3 dim Nul A = 2

2) Basis for Col(A) = $\begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix}$

Basis for Row(A) = $\begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

Basis for Nul(A) = $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{5}R_3]{} \begin{bmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free free

$$(3s-5t, s, \tfrac{3}{2}t, t, 0)$$

$$= s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

$\mathbb{R}^3 \rightarrow \mathbb{R}^6$

6) dim Nul(A) = 3 − 3 = 0

dim Row(A) = 6 − 3 = 3

rank(A^T) = 3 (row rank = col rank)

13) 7×5: $T: \mathbb{R}^5 \rightarrow \mathbb{R}^7$ largest possible rank is 5.

(rank(T) + nullity(T) = 5)

5×7: $T: \mathbb{R}^7 \rightarrow \mathbb{R}^5$ largest possible rank is 5

(rank(T) + nullity(T) = 7 but

rank(T) ≤ 5 since dim($\mathbb{R}^5$) = 5).

16) 6×4 $\quad T: \mathbb{R}^4 \to \mathbb{R}^6$

$$\text{rank}(T) + \text{nullity}(T) = 4$$

So smallest possible value of nullity is 0.

e.g. $T(a,b,c,d) = (a,b,c,d,0,0)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Additional Problems

1) Check linear ind.

$$c_1(1) + c_2(1 + x + 2x^2) + c_3(-2 + 2x^2) + c_4(1 + 2x - x^2 + x^3) = 0$$

$$c_1 + c_2 - 2c_3 + c_4 = 0$$
$$c_2 + 2c_4 = 0$$
$$2c_2 + 2c_3 - c_4 = 0$$
$$c_4 = 0$$

$$\det A = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1(1)(2) = 2 \neq 0$$

so only trivial soln. ✓

Check span. Need to show

$$c_1(1) + c_2(1 + x + 2x^2) + c_3(-2 + 2x^2) + c_4(1 + 2x - x^2 + x^3) = a + bx + cx^2 + dx^3$$

always has soln (c_1, c_2, c_3, c_4).

$$\begin{cases} c_1 + c_2 - 2c_3 + c_4 = a \\ c_2 + 2c_4 = b \\ 2c_2 + 2c_3 - c_4 = c \\ c_4 = d \end{cases}$$

but this is true since $\det A \neq 0$.

So B is a basis.

2) Show $c_1(1) + c_2 \sin x + c_3 \cos x + c_4 \sin(2x) + c_5 \cos(2x) = 0$

only has the trivial soln.

$$
\begin{aligned}
x = 0: &\quad c_1 + c_3 + c_5 = 0 \\
x = \tfrac{\pi}{4}: &\quad c_1 + \tfrac{\sqrt{2}}{2}c_2 + \tfrac{\sqrt{2}}{2}c_3 + c_4 = 0 \\
x = \tfrac{\pi}{2}: &\quad c_1 + c_2 - c_5 = 0 \\
x = \tfrac{3\pi}{4}: &\quad c_1 + \tfrac{\sqrt{2}}{2}c_2 - \tfrac{\sqrt{2}}{2}c_3 - c_4 = 0 \\
x = \pi: &\quad c_1 - c_3 + c_5 = 0
\end{aligned}
$$

$$
A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix} = 4(\sqrt{2}-1) \neq 0
$$

so $c_1 = c_2 = c_3 = c_4 = c_5 = 0$.

3) (a) not a subspace, not closed under scalar mult.

$$
\underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{\in \mathbb{Z}_{3\times 3}} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{\notin \mathbb{Z}_{3\times 3}}
$$

(b) is a subspace

$$
c_1 \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & a_3 \end{bmatrix} + c_2 \begin{bmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & b_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} c_1a_1 + c_2b_1 & & 0 \\ & c_1a_2 + c_2b_2 & \\ 0 & & c_1a_3 + c_2b_3 \end{bmatrix}
$$

$\boxed{\dim(\text{Diag}) = 3}$

Basis: $\left\{ \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix} \right\}$

spans since $\begin{bmatrix} a&0&0 \\ 0&b&0 \\ 0&0&c \end{bmatrix} = a\begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix} + b\begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix} + c\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}$

lin ind.: $c_1\begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix} + c_2\begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix} + c_3\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix} = \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$

$$\begin{bmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & c_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

so $c_1 = 0, c_2 = 0, c_3 = 0$. ✓

(c) is a subspace.

If $A, B \in \text{sym}$, $(c_1 A + c_2 B)^t$
$= c_1 A^t + c_2 B^t = c_1 A + c_2 B$.

Basis: $\left\{ \underset{M_1}{\begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}}, \underset{M_2}{\begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix}}, \underset{M_3}{\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}}, \right.$

$\boxed{\dim(\text{Sym}) = 6}$ $\left. \underset{M_4}{\begin{bmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&0 \end{bmatrix}}, \underset{M_5}{\begin{bmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{bmatrix}}, \underset{M_6}{\begin{bmatrix} 0&0&0 \\ 0&0&1 \\ 0&1&0 \end{bmatrix}} \right\}$

spans since $\begin{bmatrix} a&b&d \\ b&c&e \\ d&e&f \end{bmatrix} = aM_1 + cM_2 + fM_3 + bM_4 + dM_5 + eM_6$

lin. ind. since $c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 + c_5 M_5 + c_6 M_6 = \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$

$\Downarrow$

$$\begin{bmatrix} c_1 & c_4 & c_5 \\ c_4 & c_2 & c_6 \\ c_5 & c_6 & c_3 \end{bmatrix} = \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$$

all $c_i = 0$.

(d) Skew is a subspace.

If $A, B \in$ Skew, $(c_1 A + c_2 B)^t$
$= c_1 A^t + c_2 B^t$
$= -c_1 A - c_2 B$
$= -(c_1 A + c_2 B)$

A general matrix in Skew has the form

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

So take $B = \left\{ \underset{M_1}{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \underset{M_2}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}, \underset{M_3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}} \right\}$

as a basis.

$\boxed{\dim(\text{Skew}) = 3}$

spans since $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = aM_1 + bM_2 + cM_3$

lin. ind since $c_1 M_1 + c_2 M_2 + c_3 M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Downarrow$

$$\begin{bmatrix} 0 & c_1 & c_2 \\ -c_1 & 0 & c_3 \\ -c_2 & -c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_1 = c_2 = c_3 = 0.$

(e) not a subspace.

$$-1\underbrace{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}_{\in N}=\underbrace{\begin{bmatrix}-1&-1&-1\\-1&-1&-1\\-1&-1&-1\end{bmatrix}}_{\notin N}$$

so not closed under scalar multiplication.

(f) is a subspace

If $A, B \in TrF$, $tr(c_1A+c_2B)$

tr is a lin. transf.

$= c_1 tr(A) + c_2 tr(B)$

$= c_1(0)+c_2(0)=0$.

general matrix in TrF $\begin{bmatrix}a&c&d\\e&b&f\\g&h&-a-b\end{bmatrix}$

$$=a\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}+b\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}+c\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}+d\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$$+e\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}+f\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}+g\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}+h\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}$$

So take $B=\Big\{\overset{M_1}{\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}},\overset{M_2}{\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}},\overset{M_3}{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}},\overset{M_4}{\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}},$

$\boxed{\dim(TrF)=8}$ $\overset{M_5}{\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}},\overset{M_6}{\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}},\overset{M_7}{\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}},\overset{M_8}{\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}}\Big\}$

This spans TrF by equation above.

lin. ind since $c_1M_1+c_2M_2+\dots+c_8M_8=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

$\Downarrow$

$$\begin{bmatrix}c_1&c_3&c_4\\c_5&c_2&c_6\\c_7&c_8&-c_1-c_2\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

all $c_i=0$

(g) Inv is not a subspace. Does not contain the zero matrix.

(h) Note this is equivalent to all matrices in $M_{3\times3}$ with det = 0.

Not closed under vector addition.

$$\underset{\in NS}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}} + \underset{\in NS}{\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}} = \underset{\notin NS}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}$$

so not a subspace.

(i) is a subspace. If $A, B \in C$,

$$M = \begin{bmatrix}1&0&0\\0&2&1\\0&0&1\end{bmatrix}$$

$$\begin{aligned}(c_1A+c_2B)M &= c_1(AM)+c_2(BM)\\ &= c_1MA+c_2MB\\ &= M(c_1A+c_2B) \checkmark\end{aligned}$$

$$\begin{bmatrix}1&0&0\\0&2&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1&x_2&x_3\\x_4&x_5&x_6\\x_7&x_8&x_9\end{bmatrix} - \begin{bmatrix}x_1&x_2&x_3\\x_4&x_5&x_6\\x_7&x_8&x_9\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&1\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}x_1&x_2&x_3\\2x_4+x_7&2x_5+x_8&2x_6+x_9\\x_7&x_8&x_9\end{bmatrix} - \begin{bmatrix}x_1&2x_2&x_2+x_3\\x_4&2x_5&x_5+x_6\\x_7&2x_8&x_8+x_9\end{bmatrix}$$

$$= \begin{bmatrix}0&-x_2&-x_2\\x_4-x_7&x_8&-x_5+x_6+x_9\\0&-x_8&-x_8\end{bmatrix}$$

$x_2 = 0$

$x_4 + x_7 = 0$

$-x_5 + x_6 + x_9 = 0$

$x_8 = 0$

x_1, x_3, x_6, x_7, x_9 free

So solutionset $= (x_1, 0, x_3 - x_7, x_6 + x_9, x_6, x_7, 0, x_9)$

$$= x_1 \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} + x_3 \begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix} + x_6 \begin{bmatrix}0\\0\\0\\1\\1\\0\\0\\0\\0\end{bmatrix} + x_7 \begin{bmatrix}0\\0\\-1\\0\\0\\1\\0\\0\\0\end{bmatrix} + x_9 \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\\1\end{bmatrix}$$

$\boxed{\dim(C) = 5}$

So take

$$\mathcal{B} = \left\{ \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\-1&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} \right.$$

The above computation shows that these matrices span C.

Check lin. ind.

$$c_1 \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + c_2 \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} + c_3 \begin{bmatrix}0&0&0\\0&1&1\\0&0&0\end{bmatrix}$$

$$+ c_4 \begin{bmatrix}0&0&0\\-1&0&0\\1&0&0\end{bmatrix} + c_5 \begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}c_1&0&c_2\\-c_4&c_3+c_5&c_3\\c_4&0&c_5\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$c_1 = c_2 = c_3 = c_4 = c_5 = 0$ ✓

4) (a) is a linear transf

$T(c_1(x_1, x_2, x_3, x_4) + c_2(y_1, y_2, y_3, y_4))$

$= T(c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, c_1x_3 + c_2y_3, c_1x_4 + c_2y_4)$

$= (c_1x_4 + c_2y_4, c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, c_1x_3 + c_2y_3)$

$= c_1(x_4, x_1, x_2, x_3) + c_2(y_4, y_1, y_2, y_3)$

$= c_1T(x_1, x_2, x_3, x_4) + c_2T(y_1, y_2, y_3, y_4)$

kernel: $(0,0,0,0)$

range: all $\mathbb{R}^4$ since given any (y_1, y_2, y_3, y_4),

$T((y_2, y_3, y_4, y_1)) = (y_1, y_2, y_3, y_4)$.

so one-to-one and onto, hence bijective.

$$\boxed{T^{-1}(y_1, y_2, y_3, y_4) = (y_2, y_3, y_4, y_1)}$$

(b) $i: \mathbb{R} \to \mathbb{R}^2$ $\quad i(x) = (x, 0)$

is a linear transformation

$i(c_1x_1 + c_2x_2) = (c_1x_1 + c_2x_2, 0)$

$= c_1(x_1, 0) + c_2(x_2, 0) = c_1 i(x_1) + c_2 i(x_2)$

kernel: $x = 0$ $\qquad$ range: all $(x, 0)$, $x \in \mathbb{R}$

one-to-one, not onto, so not bijective

(c) $P(x_1, x_2, x_3) = (x_1, x_2)$

is a linear transformation

kernel: $(0, 0, x_3)$, $x_3 \in \mathbb{R}$
range: $\mathbb{R}^2$ since $P(x_1, x_2, 0) = (x_1, x_2)$

not one-to-one, but is onto - not bijective.

(d) $T(M) = AMA^{-1}$

kernel: $AMA^{-1} = 0$

$$A^{-1}AMA^{-1}A = A^{-1}0A = 0$$

$$M = 0$$

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \ker(T)$$

range$(T) = M_{3\times3}$ since for any $B \in M_{3\times3}$,

$$AMA^{-1} = B \Rightarrow M = ABA^{-1}$$

so 1-1 and onto, hence bijective.

$$\boxed{T^{-1}(B) = ABA^{-1}}$$

1e) $F(a,b) = a, b, a+b, a+2b, 2a+3b, \ldots$

So $\ker(F) = \{(0,0)\}$. one-to-one

range(F) = any Fibonacci sequence.

not onto (hence not bijective) since not every sequence is a Fibonacci sequence.

5) (a) Basis for Col(A) = $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

dim = 2

Basis for Nul(A) = $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

dim = 1.

$(-2s, -s, s)$

rank(A) + nullity(A) = dim($\mathbb{R}^3$)

$2 + 1 = 3$ ✓

(b)

$$T(M) = A \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | & | \\ Av_1 & Av_2 & Av_3 & Av_4 \\ | & | & | & | \end{pmatrix}$$

So A is multiplying each column separately.

So $\ker(T)$ = all 3×4 matrices where every column is in $\text{Null}(A)$.

So basis for $\ker(T)$:

$$\left\{ \begin{bmatrix} -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

and range(T) is all 5×4 matrices where every column is in $\text{Col}(A)$. So basis for range(T):

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

(c) $\dim \ker(T) + \dim \text{range}(T) = 4 + 8$

$= 12 = \dim(M_{3\times 4})$ ✓

6) (a) $f_1 = b_1 + b_2 x + b_3 x^2 + \dots + b_n x^{n-1}$

$f_1(x_1) = 1 \quad f_1(x_2) = 0 \, \cdots \, f_1(x_n) = 0$

$$b_1 + b_2 x_1 + b_3 x_1^2 + \dots + b_n x_1^{n-1} = 1$$
$$b_1 + b_2 x_2 + b_3 x_2^2 + \dots + b_n x_2^{n-1} = 0$$
$$\vdots$$
$$b_1 + b_2 x_n + b_3 x_n^2 + \dots + b_n x_n^{n-1} = 0$$

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ | & | & | & & | \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ | \\ 0 \end{bmatrix}}_{\vec{b}}$$

invertible so there is a unique soln $(b_1, b_2, \dots, b_n)$ hence f_1 exists and is unique.

(b) Same argument as above, but the $\vec{b}$ vector will change to the vector with 1 in the i^{th} position and 0 elsewhere for the argument for f_i.

(c) Linear Independence:

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$

Set $x = x_1$ $\quad \underbrace{c_1 f_1(x_1)}_{c_1} + c_2 f_2(x_1) + \dots + \cancel{c_n f_n(x_1)} = 0$

$$c_1 = 0$$

$x = x_2$ gives $c_2 = 0$

$\vdots$

$x = x_n$ gives $c_n = 0$.

Span: Consider $f \in P_{n-1}$. Suppose

$$f(x_1) = a_1,\ f(x_2) = a_2, \ldots, f(x_n) = a_n.$$

Then consider $p(x) = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)$

Note that $p(x_1) = a_1,\ p(x_2) = a_2, \ldots, p(x_n) = a_n$.

Since $\deg(f), \deg(p) \leq n-1$, by uniqueness of such a polynomial passing through the same n points with distinct x coordinate of $\deg \leq n-1$,

$p(x) = f(x)$.

7) (a) These polynomials are degree 2.

$f_1 = a + bx + cx^2 \quad f_1(-1) = 1 \quad f_1(0) = 0 \quad f_1(1) = 0$

$$\begin{aligned} a - b + c &= 1 \\ a &= 0 \quad \Rightarrow a = 0,\ b = -\tfrac{1}{2},\ c = \tfrac{1}{2} \\ a + b + c &= 0 \end{aligned}$$

$$\boxed{f_1(x) = -\tfrac{1}{2}x + \tfrac{1}{2}x^2}$$

$f_2 = a + bx + cx^2 \quad f_2(-1) = 0,\ f_2(0) = 1,\ f_2(1) = 0$

$$\begin{aligned} a - b + c &= 0 \\ a &= 1 \quad \Rightarrow a = 1,\ b = 0,\ c = -1 \\ a + b + c &= 0 \end{aligned}$$

$$\boxed{f_2(x) = 1 - x^2}$$

$f_3 = a + bx + cx^2 \quad f_3(-1) = 0,\ f_3(0) = 0,\ f_3(1) = 1$

$$\begin{aligned} a - b + c &= 0 \\ a &= 0 \quad \Rightarrow a = 0,\ b = \tfrac{1}{2},\ c = \tfrac{1}{2} \\ a + b + c &= 1 \end{aligned}$$

$$\boxed{f_3(x) = \tfrac{1}{2}x + \tfrac{1}{2}x^2}$$

(b) Since f_1, f_2, f_3 is a basis for P_2, f is a unique linear combination of f_1, f_2, f_3 and we must have $\boxed{f = 2f_1 + 3f_2 - f_3.}$