Section 4.1

2) (a) Yes, since if $xy \geq 0$, $(cx)(cy) = c^2(xy) \geq 0$ too.

(b) $u = (1,1)$, $v = (-2, -\frac{1}{2})$, $u+v = (-1, \frac{1}{2})$.

7) No, since not closed under scalar multiplication (e.g. $\frac{1}{2}(x^2+1) = \frac{1}{2}x^2 + \frac{1}{2}$)

20) (a) sum of two continuous functions is continuous.
continuous function times real number is continuous.

(b) Let $f, g \in C[a,b]$ with $f(a) = f(b)$, $g(a) = g(b)$.
Then $c_1 f + c_2 g \in C[a,b]$ too and

$$(c_1 f + c_2 g)(a) = c_1 f(a) + c_2 g(a)$$
$$= c_1 f(b) + c_2 g(b) = (c_1 f + c_2 g)(b) \checkmark$$

21) Yes, since

$$c_1 \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} + c_2 \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$$
$$= \begin{bmatrix} c_1 a_1 + c_2 a_2 & c_1 b_1 + c_2 b_2 \\ 0 & c_1 d_1 + c_2 d_2 \end{bmatrix} \in H.$$

31) If H is a subspace of V containing u and v, then H is closed under vector addition and scalar multiplication, so $c_1 u + c_2 v \in H$ for all reals c_1, c_2. So $\text{span}\{u, v\} \subset H$.

33) (a) To show $H+K$ is a subspace, suppose $v_1, v_2 \in H+K$.
Then $v_1 = h_1 + k_1$, $v_2 = h_2 + k_2$ for $h_i \in H$, $k_i \in K$. Then,

$$c_1 v_1 + c_2 v_2 = \underbrace{(c_1 h_1 + c_2 h_2)}_{\in H} + \underbrace{(c_1 k_1 + c_2 k_2)}_{\in K} \in H+K$$

(b) $H \subset H+K$ since $h = \underbrace{h}_{\in H} + \underbrace{0}_{\in K}$ for all $h \in H$ and $k = \overset{\in H}{0} + \overset{\in K}{k}$ for all $k \in K$

Since H and K and $H+K$ are all vector spaces,
H and K are subspaces of $H+K$.

Section 4.1

Section 4.2

3) $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix}$ $R_1 - 3R_2$

$$\mathrm{Nul}(A) = \{ (7s-6t, -4s+2t, s, t) \}$$
$$= \left\{ s\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

12) not a vector space since it does not contain $(0,0,0,0)$.

14)
$$\underbrace{\begin{bmatrix} -a+2b \\ a-2b \\ 3a-6b \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix}}$$

$\mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix} \right\}$ is a subspace since the span of vectors is always a subspace.

22) $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$ nonzero vector in $\mathrm{Nul}(A)$ is $\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}$

nonzero vector in $\mathrm{Col}(A) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

33) $T: M_{2\times 2} \rightarrow M_{2\times 2}$
$T(A) = A + A^t$

(a) $T(c_1 A + c_2 B) = (c_1 A + c_2 B) + (c_1 A + c_2 B)^t$
$= c_1 (A + A^t) + c_2 (B + B^t)$
$= c_1 T(A) + c_2 T(B)$. ✓

(b) $B = B^t$ $\boxed{A = \frac{1}{2}B}$ since $T(\frac{1}{2}B) = (\frac{1}{2}B) + (\frac{1}{2}B)^t$
$= \frac{1}{2}B + \frac{1}{2}B = B$.

(c) For any A, $(A+A^t)^t = A^t + A = A + A^t$. (b) shows that symmetric 2×2 matrices are a subset of range(T) and $T(A)$ is always a symmetric 2×2 matrix. So range(T) is all symmetric 2×2 matrices.

(d) $\ker(T)$ = all skew-symmetric matrices $(A = -A^t)$

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

Additional Problems

1) (a) Yes. If $p(x)$ and $q(x)$ are polynomials with constant term 0. Then $c_1p(x)+c_2q(x)$ also has constant term 0.

(b) No. In P_3 for example, $x^3, x^2+x+1 \in V_3$ but

$$x^3+x^2+x+1 \notin V_3.$$

(c) Yes. If $p(x)$ and $q(x)$ are even, then

$$(c_1p+c_2q)(-x) = c_1p(-x)+c_2q(-x)$$
$$= c_1p(x)+c_2q(x) = (c_1p+c_2q)(x).$$

So c_1p+c_2q is also even.

(d) Yes. Suppose $p(x), q(x) \in W_n$, then if $a_0, a_1, \dots, a_n$ and $b_0, b_1, \dots, b_n$ are the coefficients of p and q respectively, then $a_0+a_1+\dots+a_n=0$ and $b_0+b_1+\dots+b_n=0$.

The coefficients of $c_1p(x)+c_2q(x)$ are $c_1a_0+c_2b_0, c_1a_1+c_2b_1, c_1a_2+c_2b_2, \dots, c_1a_n+c_2b_n$ and indeed,

$$(c_1a_0+c_2b_0)+(c_1a_1+c_2b_1)+\dots+(c_1a_n+c_2b_n)$$
$$= c_1(a_0+a_1+\dots+a_n)+c_2(b_0+b_1+\dots+b_n)$$
$$= c_1(0)+c_2(0)=0.$$

(e) $\ker\left(\frac{d}{dx}\right) = P_0$ (all constant polynomials)

$\text{range}\left(\frac{d}{dx}\right) = P_{n-1}$ (since given $f \in P_{n-1}$, $\frac{d}{dx}\left(\underbrace{\int_0^x f(t)\,dt}_{\in P_n}\right) = f(x)$)

(f) $\ker(E_0) = U_n$ (polynomials with constant term 0)

$\text{range}(E_0) = \mathbb{R}$ (the constant polynomial $f(x)=a$ maps to a)

2) (a) det : $M_{3\times3}\to\mathbb{R}$ is not a linear transformation.

$$\underset{\overset{\uparrow}{8}}{\det\left(2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)} \neq \underset{\overset{\uparrow}{2\cdot1=2}}{2\det\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)}$$

(b) $t: M_{3\times2}\to M_{2\times3}$ is a linear transformation.

$$t(c_1A+c_2B) = c_1A^t+c_2B^t = c_1t(A)+c_2t(B)$$

$\ker(t)=\left\{\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}\right\}$ range$(t)=M_{2\times3}$

since given any $\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}\in M_{2\times3}$,

$$t\left(\begin{bmatrix}a&d\\b&e\\c&f\end{bmatrix}\right)=\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}.$$

(c) $F:\mathbb{R}^4\to\mathbb{R}^2$ is a linear transformation.

$$F\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right)=\underset{\overset{\uparrow}{A}}{\begin{bmatrix}1&2&0&1\\0&1&0&1\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$\begin{aligned}F(c_1v_1+c_2v_2)&=A(c_1v_1+c_2v_2)\\&=c_1Av_1+c_2Av_2=c_1F(v_1)+c_2F(v_2)\end{aligned}$$

$$\begin{bmatrix}1&2&0&1\\0&1&0&1\end{bmatrix}\to\begin{bmatrix}1&0&0&-1\\0&1&0&1\end{bmatrix}^{R_1-2R_2}$$

$$\ker(F)=\{(t,-t,s,t)\}=\left\{s\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right\}$$

$$\text{range}(F)=\text{col}(A)=\mathbb{R}^2$$

(d) $P: \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation.

$P(x_1, x_2, x_3, x_4) = (x_4, x_1, x_2, x_3)$

$P(c_1(x_1, x_2, x_3, x_4) + c_2(y_1, y_2, y_3, y_4))$

$= (c_1x_4 + c_2y_4, c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, c_1x_3 + c_2y_3)$

$= c_1(x_4, x_1, x_2, x_3) + c_2(y_4, y_1, y_2, y_3)$

$= c_1 P(x_1, x_2, x_3, x_4) + c_2 P(y_1, y_2, y_3, y_4)$

(e) $M: \mathbb{R}^2 \to \mathbb{R}$ is not a linear transformation.

$\underbrace{M(2(1,1))}_{2\cdot 2 = 4} \neq \underbrace{2M(1,1)}_{2(1\cdot 1)=2}$

(f) $D: C(\mathbb{R}) \to C(\mathbb{R})$ is a linear transformation.

$D(c_1f + c_2g) = \dfrac{c_1f(x) + c_2g(x)}{x^2+1}$

$= c_1\left(\dfrac{f(x)}{x^2+1}\right) + c_2\left(\dfrac{g(x)}{x^2+1}\right)$

$= c_1 D(f) + c_2 D(g)$

$\ker(D) = \{f(x) = 0\}$ $\quad \dfrac{f(x)}{x^2+1} = 0 \Rightarrow f(x) = 0$

$\text{range}(D) = C(\mathbb{R})$ since given any function $f \in C(\mathbb{R})$,

$D(f(x)\cdot(x^2+1)) = f(x)$.

3) (a) $L(V, \mathbb{R})$ is the set of all linear transformations from V to $\mathbb{R}$.
The vectors are linear transformations.

To add $T_1, T_2: V \to \mathbb{R}$, let $(T_1 + T_2)(v) = T_1(v) + T_2(v)$
and to scalar multiply a linear transformation T by a real number c, define $(cT)(v) = c\cdot T(v)$.

① Check that $L(V, \mathbb{R})$ is closed under addition. So check $T_1 + T_2$ is also a linear transformation from V to $\mathbb{R}$.

Check linearity: $(T_1 + T_2)(c_1v_1 + c_2v_2) = T_1(c_1v_1 + c_2v_2) + T_2(c_1v_1 + c_2v_2)$

$= c_1T_1(v_1) + c_2T_1(v_2) + c_1T_2(v_1) + c_2T_2(v_2)$

$= c_1(T_1 + T_2)(v_1) + c_2(T_1 + T_2)(v_2)$

② Check $L(V,\mathbb{R})$ is closed under scalar multiplication.
Check that $cT: V \to \mathbb{R}$ is a linear transformation too.

$$\begin{aligned}(cT)(c_1v_1+c_2v_2) &= c\,T(c_1v_1+c_2v_2)\\ &= cc_1T(v_1)+cc_2T(v_2)\\ &= c_1(cT)(v_1)+c_2(cT)(v_2).\ \checkmark\end{aligned}$$

③ The zero vector is the zero linear transformation $Z: V \to \mathbb{R}$ where $Z(v)=0$ for all $v \in V$.

④ To take the additive inverse of $T: V \to \mathbb{R}$ define

$$(-T)(v) = -1 \cdot T(v).$$

Other properties are easy to check.

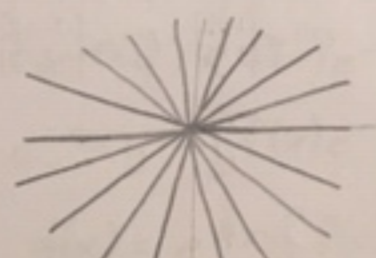
(b) If $T: \mathbb{R} \to \mathbb{R}$ is a linear transformation,

$$T(x) = T(x \cdot 1) \underset{\substack{\uparrow \\ \text{linearity}}}{=} xT(1) = kx \quad \text{where } k = T(1).$$

Since $T(1)=k$ can be any real number, any linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is of the form

$$T(x) = kx \quad \text{where } k \text{ is any real number.}$$

Considering these as functions from $\mathbb{R}$ to $\mathbb{R}$, $L(\mathbb{R},\mathbb{R})$ is thus the set of all non-vertical lines that pass through the origin.



4)(a) Check linearity.

$$\begin{aligned}\left(\frac{d}{dx}-4\right)(c_1f+c_2g) &= \frac{d}{dx}(c_1f+c_2g)-4(c_1f+c_2g)\\ &= c_1f'(x)+c_2g'(x)-4c_1f(x)-4c_2g(x)\\ &= (c_1f'-4c_1f)+(c_2g'-4c_2g)\\ &= c_1(f'-4f)+c_2(g'-4g)\\ &= c_1\left(\frac{d}{dx}-4\right)(f)+c_2\left(\frac{d}{dx}-4\right)(g)\end{aligned}$$

(b) $\left(\frac{d}{dx}-4\right)(f)=0$

$$f'-4f=0$$

$$y'-4y=0$$

$$\frac{dy}{dx}=4y \quad \int \frac{1}{y}\,dy=\int 4\,dx$$

$$\ln|y|=4x+C$$

$$|y|=e^{C}e^{4x} \quad y=\pm Ae^{4x}$$

$$\ker\left(\frac{d}{dx}-4\right)=\{Ae^{4x} \text{ where } A \text{ is any real number}\}$$

We claim that range$\left(\frac{d}{dx}-4\right)=C(\mathbb{R})$.

Given any $f\in C(\mathbb{R})$, there exists a function $g\in C'(\mathbb{R})$ such that $\left(\frac{d}{dx}-4\right)(g)=f$. To show this, we can solve for g in terms of f.

$$y'-4y=f(x) \qquad I(x)=e^{\int -4\,dx}=e^{-4x}$$

$$e^{-4x}y'+(-4e^{-4x})y=f(x)e^{-4x}$$

$$\frac{d}{dx}(e^{-4x}y)=f(x)e^{-4x}$$

$$e^{-4x}y=\int f(x)e^{-4x}\,dx$$

$$g(x)=e^{4x}\int f(x)e^{-4x}\,dx$$