Problem Set 2 Solutions

Section 1.7 2, 8, 10, 34, 36, 38

2) $\begin{bmatrix}0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\5\\-8\end{bmatrix}, \begin{bmatrix}-3\\4\\1\end{bmatrix}$ linear independent since

$$A = \begin{bmatrix}0 & 0 & -3\\0 & 5 & 4\\2 & -8 & 1\end{bmatrix} \to \begin{bmatrix}2 & -8 & 1\\0 & 5 & 4\\0 & 0 & -3\end{bmatrix}$$

$$\to \begin{bmatrix}2 & -8 & 0\\0 & 5 & 0\\0 & 0 & 3\end{bmatrix} \begin{matrix}R_1+\frac{1}{3}R_3\\R_2+\frac{4}{3}R_3\end{matrix} \to \begin{bmatrix}2 & 0 & 0\\0 & 5 & 0\\0 & 0 & 3\end{bmatrix} R_1+\tfrac{8}{5}R_2 \to \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \begin{matrix}\frac{1}{2}R_1\\\frac{1}{5}R_2\\\frac{1}{3}R_3\end{matrix}$$

so A has reduced row echelon form that is the identity so there are no free columns.

8) $$\begin{bmatrix}1 & -3 & 3 & -2\\-3 & 7 & -1 & 2\\0 & 1 & -4 & 3\end{bmatrix} \to \begin{bmatrix}1 & -3 & 3 & -2\\0 & -2 & 8 & -4\\0 & 1 & -4 & 3\end{bmatrix} R_2+3R_1$$

$$\to \begin{bmatrix}1 & -3 & 3 & -2\\0 & 0 & 0 & 2\\0 & 1 & -4 & 3\end{bmatrix} R_2+2R_3 \to \begin{bmatrix}1 & -3 & 3 & -2\\0 & 1 & -4 & 3\\0 & 0 & 0 & 2\end{bmatrix} \updownarrow$$

$$\to \begin{bmatrix}1 & -3 & 3 & -2\\0 & 1 & -4 & 3\\0 & 0 & 0 & 1\end{bmatrix} \tfrac{1}{2}R_3 \to \begin{bmatrix}1 & -3 & 3 & 0\\0 & 1 & -4 & 0\\0 & 0 & 0 & 1\end{bmatrix} \begin{matrix}R_1+2R_3\\R_2-3R_3\end{matrix}$$

There is a free column so columns are not linearly independent. Or four vectors can never be linearly independent in $\mathbb{R}^3$.

10) (a) When does $\begin{bmatrix}1 & -3\\-3 & 9\\2 & -6\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}5\\-7\\h\end{bmatrix}$ have a solution?

$$\left[\begin{array}{cc|c}1 & -3 & 5\\-3 & 9 & -7\\2 & -6 & h\end{array}\right] \to \left[\begin{array}{cc|c}1 & -3 & 5\\0 & 0 & 8\\0 & 0 & h-10\end{array}\right] \begin{matrix}R_2+3R_1\\R_3-2R_1\end{matrix} \quad \boxed{h=10}$$

I solved 9) instead of 10)
10) is solved the same way.
The answer is for 10a) no possible h
and 10b) all h

(b) When does $\begin{bmatrix}1 & -3 & 5\\-3 & 9 & -7\\2 & -6 & h\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ have a nontrivial solution?

$$\begin{bmatrix}1 & -3 & 5\\-3 & 9 & -7\\2 & -6 & h\end{bmatrix} \to \begin{bmatrix}1 & -3 & 5\\0 & 0 & 8\\0 & 0 & h-10\end{bmatrix} \begin{matrix}R_2+3R_1\\R_3-2R_1\end{matrix} \to \begin{bmatrix}1 & -3 & 5\\0 & 0 & 1\\0 & 0 & h-10\end{bmatrix} \tfrac{1}{8}R_2 \to \begin{bmatrix}1 & -3 & 0\\0 & 0 & 1\\0 & 0 & h-10\end{bmatrix} R_1-5R_2$$

has a nontrivial solution $\boxed{\text{for all } h}$

34) True, since if $v_3 = 0$, then $0v_1 + 0v_2 + 1v_3 + 0v_4 = 0$.

36) False, $v_1 = (1,1,1,1)$, $v_2 = (1,1,1,1)$, $v_3 = (1,1,1,0)$, $v_4 = (1,1,1,1)$

38) True. If $v_1, \ldots, v_4$ are linearly independent, then $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$ only has the solution $c_1 = c_2 = c_3 = c_4 = 0$. We claim v_1, v_2, v_3 are also linearly independent. If not, then $c_1v_1 + c_2v_2 + c_3v_3 = 0$ for c_1, c_2, c_3 not all zero. But then, $c_1v_1 + c_2v_2 + c_3v_3 + 0v_4 = 0$ for c_1, c_2, c_3 not all zero, which contradicts the linear independence of v_1, v_2, v_3, v_4. So by contradiction, v_1, v_2, v_3 are linearly independent too.

Section 2.1 1, 8, 9, 10, 27

1) $-2A = -2\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}$

$B - 2A = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} - 2\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}$

$AC =$ undefined since A is 2×3 and C is 2×2

$CD = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$

8) If BC is 3×4, then B has <u>3 rows.</u>

9) $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

$\begin{bmatrix} 23 & 5k-10 \\ -9 & k+15 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ -3k+6 & k+15 \end{bmatrix}$ need $5k-10=15$, $-3k+6=-9$

$\boxed{k=5}$

10) $AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$

but $B \neq C$

$AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$

27) $u = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$ $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $u^Tv = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = [-2a+3b$

$v^Tu = \begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -2a+3b-4c \end{bmatrix}$

$uv^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$

$vu^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} -2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$

Section 2.2 3, 6, 14, 22, 32

3) $\left[\begin{array}{cc|cc} 8 & 5 & 1 & 0 \\ -7 & -5 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ -7 & -5 & 0 & 1 \end{array}\right] R_1 + R_2$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & -5 & 7 & 8 \end{array}\right] R_2 + 7R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & -\frac{7}{5} & -\frac{8}{5} \end{array}\right] -\frac{1}{5} R_2$

$$\boxed{\begin{bmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{bmatrix}}$$

6) $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$\boxed{x_1 = 2,\ x_2 = -5}$$

14) $(B-C)D = 0 \Rightarrow (B-C)DD^{-1} = OD^{-1} \Rightarrow (B-C)I = 0 \Rightarrow B-C=0 \Downarrow B=C$

22) When the columns of an nxn matrix span $\mathbb{R}^n$, there is a pivot in every row of the reduced row echelon form. Since A is square, the reduced row echelon form of A is the identity, which implies that A is invertible.

32) $\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 4R_1 \\ R_3 + 2R_1 \end{array}$

$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -7 & 2 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right] \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 2R_2 \end{array}$ $\boxed{\text{The matrix is not invertible.}}$

$$[C,A] = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -8 & -1 \end{bmatrix}$$

$$[B,[C,A]] = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ -8 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ -8 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -11 \\ 17 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 7 \\ -25 & 10 \end{bmatrix} = \begin{bmatrix} 12 & -18 \\ 42 & -12 \end{bmatrix}$$

$$[A,B] = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & -1 \end{bmatrix}$$

$$[C,[A,B]] = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 8 \end{bmatrix} - \begin{bmatrix} 8 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -2 \\ -4 & 7 \end{bmatrix}$$

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$$

$$= \begin{bmatrix} -5 & 20 \\ -38 & 5 \end{bmatrix} + \begin{bmatrix} 12 & -18 \\ 42 & -12 \end{bmatrix} + \begin{bmatrix} -7 & -2 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

Or another approach...

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$$

$$= [A, BC-CB] + [B, CA-AC] + [C, AB-BA]$$

$$= A(BC-CB) - (BC-CB)A + B(CA-AC) - (CA-AC)B + C(AB-BA) - (AB-BA)$$

$$= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BA$$

$= 0$ (terms cancel pairwise)

2) (a) $\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix}$ since $\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} + a_{n-2} \\ a_{n-1} \end{pmatrix}$

so $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) $F^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad F^4 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$

$F^8 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 34 & 21 \\ 21 & 13 \end{pmatrix}$ $\quad F^{16} = \begin{pmatrix} 34 & 21 \\ 21 & 13 \end{pmatrix}\begin{pmatrix} 34 & 21 \\ 21 & 13 \end{pmatrix} = \begin{pmatrix} 1597 & 987 \\ 987 & 610 \end{pmatrix}$

$F^{32} = \begin{pmatrix} 1597 & 987 \\ 987 & 610 \end{pmatrix}\begin{pmatrix} 1597 & 987 \\ 987 & 610 \end{pmatrix} = \begin{pmatrix} 3524578 & 2178309 \\ 2178309 & 1346269 \end{pmatrix}$

So $\begin{pmatrix} a_{33} \\ a_{32} \end{pmatrix} = \begin{pmatrix} 3524578 & 2178309 \\ 2178309 & 1346269 \end{pmatrix}\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$

$$\boxed{\begin{aligned} a_{33} &= 3524578a_1 + 2178309a_0 \\ a_{32} &= 2178309a_1 + 1346269a_0 \end{aligned}}$$

3) (a) If two of the a_i are the same, then two rows of the Vandermonde matrix are identical. But a matrix with two identical rows is not invertible.

(b) Let $f(x) = b_1 + b_2x + \dots + b_{n-1}x^{n-1}$ be a polynomial with n distinct roots, $x_1, x_2, \dots, x_n$. Then,

$$\begin{aligned} b_1 + b_2x_1 + \dots + b_{n-1}x_1^{n-1} &= 0 \\ b_1 + b_2x_2 + \dots + b_{n-1}x_2^{n-1} &= 0 \\ &\vdots \\ b_1 + b_2x_n + \dots + b_{n-1}x_n^{n-1} &= 0 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{0}$$

But the coefficient matrix A is a Vandermonde matrix, and since $x_1, x_2, \dots, x_n$ are distinct, A is invertible. So by the Invertible Matrix Theorem, the only solution is $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

so $f(x) = 0$.

4) $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_1 + R_2$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{array}\right] R_3 - 2R_1$ $\boxed{A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}}$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\boxed{D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$

$(A^{-1}DA)^{2019} = A^{-1}D^{2019}A$

$(A^{-1}DA)^2 = A^{-1}D\underbrace{AA^{-1}}_{I}DA = A^{-1}D^2A$

$(A^{-1}DA)^3 = A^{-1}D\underbrace{AA^{-1}}_{I}D\underbrace{AA^{-1}}_{I}DA = A^{-1}D^3A$

$= A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2019} A$

$= A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ -4 & 4 & -1 \end{bmatrix}}$