Problem Set 1 Solutions

Section 1.1

11) $$\left[\begin{array}{ccc|c} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{array}\right] R_3-3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{array}\right] R_3+2R_2 \to \left[\begin{array}{ccc|c} 1 & 0 & -7 & 13 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1-3R_2 \\ \\ \frac{1}{2}R_3 \end{array}$$

$\boxed{\text{inconsistent (no solutions)}}$

14) $$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{array}\right] R_2+R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{array}\right] R_3+2R_2 \to \left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \tfrac{1}{7}R_3$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] R_2-R_3 \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] R_1+3R_2$$

$\boxed{(x_1,x_2,x_3)=(2,-1,1)}$

18) $$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{array}\right] R_3-R_1 \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right] R_3-R_2$$

$\boxed{\text{No, no solution.}}$

21) $$\left[\begin{array}{cc|c} 1 & 3 & -2 \\ -4 & h & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & -2 \\ -4 & h & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & h+12 & 0 \end{array}\right] R_2+4R_1$$

would be consistent if $h+12=0$ $(0=0)$ and also would be consistent if $h+12\neq 0$ (would have two pivots). $\boxed{\text{So } h \text{ can be any real number.}}$

23) (a) True, e.g. $R_1 \leftrightarrow R_2$ reversed by $R_1 \leftrightarrow R_2$,
$R_1 + 2R_2$ reversed by $R_1 - 2R_2$, cR_1 for $c \neq 0$ reversed by $\frac{1}{c}R_1$.
(b) False, since a 5×6 matrix has 5 rows.
(c) True, definition of solution.
(d) True.

24) (a) True (b) False, two matrices are row equivalent if they can be obtained from each other by elementary row operations.
(c) False, inconsistent means no solution.
(d) True

Section 1.2

3) $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \begin{matrix} \\ R_2 - 4R_1 \\ R_3 - 6R_1 \end{matrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} \\ -\frac{1}{3}R_2 \\ -\frac{1}{5}R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 - R_2 \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 2R_2$$

pivot columns

12) $$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right] R_3 + R_1$$

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_3 + 4R_2$$

x_2, x_4 free
$x_1 - 7x_2 + 6x_4 = 5$
$x_3 - 2x_4 = -3$

$x_2 = s$ $\quad x_4 = t$ $\quad x_3 = -3 + 2t$
$x_1 = 7s - 6t + 5$

Solution set: $\boxed{(5 + 7s - 6t, s, -3 + 2t, t) \text{ for } s, t \text{ real}}$

13) $\left[\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & -12 & 1 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $R_1 + 3R_2$

$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $R_1 + R_3$ $\quad x_3, x_5$ free

$x_3 = s,\ x_5 = t$

$x_1 = 5 + 3t,\ x_2 = 1 + 4t,\ x_4 = 4 - 9t$

$\boxed{\text{Solution set: } (5+3t, 1+4t, s, 4-9t, t) \text{ for reals } s, t}$

20) $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$ $R_2 - 3R_1$

(a) For no solution, let $\boxed{h = 9, k = 7}$ $\quad \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 1 \end{array}\right]$

(b) For a unique solution, let $\boxed{h = 10, k = 6}$ $\quad \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 0 \end{array}\right]$

(c) For many solutions, let $\boxed{h = 9, k = 6}$ $\quad \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 0 \end{array}\right]$

23) $3 \overbrace{\left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \end{array}\right]}^{5}$

The coefficient matrix is 3×5 and has three pivot columns, so each row has a pivot. So the system is consistent.

29) Since every row has a pivot, but the coefficient matrix has less rows than columns, this means there is at least one free column. So there are infinitely many solutions.

30) $\begin{cases} x + y + z = 2 \\ x + y + z = 3 \end{cases}$

31) Yes. $\begin{cases} x + y = 1 \\ x - 2y = 2 \\ 2x - y = 3 \end{cases}$ is consistent.

Section 1.3

11) Need to solve

$$c_1\begin{bmatrix}1\\-2\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\2\end{bmatrix} + c_3\begin{bmatrix}5\\-6\\8\end{bmatrix} = \begin{bmatrix}2\\-1\\6\end{bmatrix}$$

$$\begin{bmatrix}c_1+5c_3\\-2c_1+c_2-6c_3\\2c_2+8c_3\end{bmatrix} = \begin{bmatrix}2\\-1\\6\end{bmatrix}$$

$$\begin{cases} c_1+5c_3=2\\ -2c_1+c_2-6c_3=-1\\ 2c_2+8c_3=6\end{cases}$$

$$\left[\begin{array}{ccc|c}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&5&2\\0&1&4&3\\0&2&8&6\end{array}\right] R_2+2R_1 \to \left[\begin{array}{ccc|c}1&0&5&2\\0&1&4&3\\0&0&0&0\end{array}\right] R_3-2R_2$$

So there is a solution c_1, c_2, c_3. $\boxed{\text{So yes, } b \text{ is a linear combination of } a_1, a_2, a_3.}$

18) For what h does

$$c_1\begin{bmatrix}1\\0\\-2\end{bmatrix} + c_2\begin{bmatrix}-3\\1\\8\end{bmatrix} = \begin{bmatrix}h\\-5\\-3\end{bmatrix} \Leftrightarrow \begin{matrix}c_1-3c_2=h\\ c_2=-5\\ -2c_1+8c_2=-3\end{matrix}$$

have a solution?

$$\left[\begin{array}{cc|c}1&-3&h\\0&1&-5\\-2&8&-3\end{array}\right] \to \left[\begin{array}{cc|c}1&-3&h\\0&1&-5\\0&2&-3+2h\end{array}\right] R_3+2R_1 \to \left[\begin{array}{cc|c}1&0&h-15\\0&1&-5\\0&0&7+2h\end{array}\right] \begin{matrix}R_1+3R_2\\ \\ R_3-2R_2\end{matrix}$$

need $7+2h=0 \Rightarrow \boxed{h=-\frac{7}{2}}$

21) Need to show $c_1\begin{bmatrix}2\\-1\end{bmatrix} + c_2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}h\\k\end{bmatrix} \Leftrightarrow \begin{matrix}2c_1+2c_2=h\\ -c_1+c_2=k\end{matrix}$

always has a solution.

$$\left[\begin{array}{cc|c}2&2&h\\-1&1&k\end{array}\right] \to \left[\begin{array}{cc|c}1&1&\frac{h}{2}\\0&2&\frac{h}{2}+k\end{array}\right] \begin{matrix}\frac{1}{2}R_1\\ R_2+\frac{1}{2}R_1\end{matrix}$$

$$\left[\begin{array}{cc|c}1&1&\frac{h}{2}\\0&1&\frac{h}{4}+\frac{k}{2}\end{array}\right] \tfrac{1}{2}R_2 \to \left[\begin{array}{cc|c}1&0&\frac{h}{4}-\frac{k}{2}\\0&1&\frac{h}{4}+\frac{k}{2}\end{array}\right] R_1-R_2$$

always has a solution for (c_1, c_2) since there are no free columns.

22) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ since span (cols of A) is $\{(s,s,s)$ for s real$\}$.

24) a. True b. $(u-v)+v=u$. True. c. False d. True (also contains entire plane spanned by u and v)

e. True.

25) a. $\{a_1, a_2, a_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} \right\}$

(the actual set of three objects).

b is not in $\{a_1, a_2, a_3\}$ which only has three vectors.

b. $W = \text{span}\{a_1, a_2, a_3\}$ has infinitely many vectors

$c_1(1,0,-2) + c_2(0,3,6) + c_3(-4,-2,3)$.

Need to solve

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{array}\right] R_3 + 2R_1$$

$$\swarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right] R_3 - 2R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} R_1 - 4R_3 \\ R_2 - 2R_3 \\ -R_3 \end{array}$$

$$\swarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \tfrac{1}{3}R_2$$

$\boxed{\text{has a solution so } b \text{ is in } W = \text{span}\{a_1, a_2, a_3\}}$

c. $a_1 = 1a_1 + 0a_2 + 0a_3$

Section 1.5

7) $\left[\begin{array}{cccc|c} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$ $R_1 - 3R_2$

$x_3 = s \quad x_4 = t$ (free)

$$\boxed{\begin{aligned} &(-9s+8t, 4s-5t, s, t) \\ &= s\begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix} \end{aligned}}$$

12) $\left[\begin{array}{cccccc|c} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 5 & 2 & -6 & 9 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $R_2 + 8R_3$

$\left[\begin{array}{cccccc|c} 1 & 5 & 0 & 8 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $R_1 - 2R_2$

$x_2 = s$
$x_4 = t$ (free)
$x_5 = u$

$$\boxed{\begin{aligned} &(-5s-8t-u, s, 7t-4u, t, u, 0) \\ &= s\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}}$$

21) $p = (2,-5)$ on line, $q - p = (-5, 6)$ gives direction

$$\boxed{(2,-5) + s(-5,6) = (2-5s, -5+6s)}$$

23) a. True, always has zero solution.

b. True

c. False, iff it has no free variables.

d. True.

e. True

26) Solution to $Ax = b$ is $\underbrace{y_p}_{\text{part. soln}} + \underbrace{y_{hom}}_{\text{hom. soln.}}$. But if $Ax = 0$ only has the trivial (zero) solution, y_{hom} is just the zero vector so that the only solution to $Ax = b$ is the single (hence unique) particular solution.

28) No, since if solution set is a plane through the origin, then $(0, 0, \ldots, 0)$ is a solution. But $(0, 0, \ldots, 0)$ cannot be a solution of $Ax = b$ since $b \neq (0, 0, \ldots, 0)$.

30) (a) Yes, since there is one free column.

(b) No, since A has a row of zeros. (So can get $0 \neq 1$ or something like that)

32) (a) Yes, since there are two free columns.

(b) Yes, since every row has a pivot.

Additional Problems

1) a) Need to check $2(c_1 r_1 + c_2 s_1) + 3(c_1 r_2 + c_2 s_2) + (c_1 r_3 + c_2 s_3) = 0$

But $2r_1 + 3r_2 + r_3 = 0$ and $2s_1 + 3s_2 + s_3 = 0$ since $(r_1, r_2, r_3), (s_1, s_2, s_3)$ are solutions.

So $2(c_1 r_1 + c_2 s_1) + 3(c_1 r_2 + c_2 s_2) + (c_1 r_3 + c_2 s_3)$

$= c_1(2r_1 + 3r_2 + r_3) + c_2(2s_1 + 3s_2 + s_3) = c_1 \cdot 0 + c_2 \cdot 0 =$

Can check other equations in the same way.

b) Need to check $a_{11}(c_1 r_1 + c_2 s_1) + a_{12}(c_1 r_2 + c_2 s_2) + \ldots + a_{1n}(c_1 r_n + c_2 s_n) = 0$

But $a_{11} r_1 + a_{12} r_2 + \ldots + a_{1n} r_n = 0$, $a_{11} s_1 + a_{12} s_2 + \ldots + a_{1n} s_n = 0$ since $(r_1, r_2, \ldots, r_n), (s_1, s_2, \ldots, s_n)$ are solutions.

So $a_{11}(c_1 r_1 + c_2 s_1) + a_{12}(c_1 r_2 + c_2 s_2) + \ldots + a_{1n}(c_1 r_n + c_2 s_n)$

$= c_1(a_{11} r_1 + a_{12} r_2 + \ldots + a_{1n} r_n) + c_2(a_{11} s_1 + a_{12} s_2 + \ldots + a_{1n} s_n)$

$= c_1 \cdot 0 + c_2 \cdot 0 = 0$ ✓

Can check other equations in the same way.