Problem Set 11 Solutions

Section 9.5

12) $\begin{bmatrix} 1 & 3 \\ 12 & 1 \end{bmatrix}$ $\quad \begin{vmatrix} 1-x & 3 \\ 12 & 1-x \end{vmatrix} = x^2 - 2x - 35$

$$= (x-7)(x+5)$$

$\lambda = 7 \quad \begin{bmatrix} -6 & 3 \\ 12 & -6 \end{bmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = -5 \quad \begin{bmatrix} 6 & 3 \\ 12 & 6 \end{bmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\boxed{\vec{x}(t) = C_1 e^{7t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

15) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

$$\begin{vmatrix} 1-x & 2 & 3 \\ 0 & 1-x & 0 \\ 2 & 1 & 2-x \end{vmatrix} = (1-x)(1-x)(2-x) + 2(3)(x-1)$$

$$= (x-1)((x-1)(2-x)+6)$$

$$= (x-1)(-x^2+3x+4)$$

$$= -(x-1)(x-4)(x+1)$$

$\lambda = -1 \quad \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix} \quad v_1 = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$

$\lambda = 1 \quad \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix}$

$\lambda = 4 \quad \begin{bmatrix} -3 & 2 & 3 \\ 0 & -3 & 0 \\ 2 & 1 & -2 \end{bmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\boxed{\vec{x}(t) = C_1 e^{-t} \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} -1 \\ 6 \\ -4 \end{pmatrix} + C_3 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

Section 9.6

6) $\begin{bmatrix} -2 & -2 \\ 4 & 2 \end{bmatrix}$ $\quad \begin{vmatrix} -2-x & -2 \\ 4 & 2-x \end{vmatrix}$

$= -4 + x^2 + 8 = x^2 + 4$

$\lambda = \pm 2i$

$\lambda = 2i \quad \begin{bmatrix} -2-2i & -2 \\ 4 & 2-2i \end{bmatrix} \quad v_1 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$

$e^{2it}\begin{pmatrix} 1 \\ -1-i \end{pmatrix} = (\cos(2t) + i\sin(2t))\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$

$= \begin{pmatrix} \cos(2t) \\ -\cos(2t)+\sin(2t) \end{pmatrix} + i\begin{pmatrix} \sin(2t) \\ -\cos(2t)-\sin(2t) \end{pmatrix}$

$$\boxed{\vec{x}(t) = C_1\begin{pmatrix} \cos(2t) \\ -\cos(2t)+\sin(2t) \end{pmatrix} + C_2\begin{pmatrix} \sin(2t) \\ -\cos(2t)-\sin(2t) \end{pmatrix}}$$

13) $\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$ $\quad \begin{vmatrix} -3-x & -1 \\ 2 & -1-x \end{vmatrix}$

$= 3 + 4x + x^2 + 2 = x^2 + 4x + 5$

$\lambda = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$

$\lambda = -2+i \quad \begin{bmatrix} -1-i & -1 \\ 2 & 1-i \end{bmatrix} \quad v_1 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$

$e^{-2t}e^{it}\begin{pmatrix} 1 \\ -1-i \end{pmatrix} = e^{-2t}(\cos t + i\sin t)\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$

$= e^{-2t}\begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix} + ie^{-2t}\begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix}$

$\vec{x}(t) = C_1 e^{-2t}\begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix}$

(a) $\vec{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\vec{x}(0) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$C_1 = -1 \quad C_2 = 1$$

$$\boxed{\vec{x}(t) = -e^{-2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix} + e^{-2t} \begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix}}$$

(b) $\vec{x}(\pi) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\vec{x}(\pi) = C_1 e^{-2\pi} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-2\pi} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C_1 = -e^{2\pi}, \; C_2 = 0$$

$$\boxed{\vec{x}(t) = -e^{2\pi - 2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix}}$$

(c) $\vec{x}(-2\pi) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\vec{x}(2\pi) = C_1 e^{4\pi} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4\pi} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$C_1 = 2e^{-4\pi} \quad C_2 = -3e^{-4\pi}$$

$$\boxed{\vec{x}(t) = 2e^{-4\pi - 2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix} - 3e^{-4\pi - 2t} \begin{pmatrix} \sin t \\ -\cos t - \sin t \end{pmatrix}}$$

(d) $\vec{x}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x}\left(\frac{\pi}{2}\right) = C_1 e^{-\pi} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 e^{-\pi} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$C_1 = e^{\pi} \quad C_2 = 0$$

$$\boxed{\vec{x}(t) = e^{\pi - 2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix}}$$

Non-textbook:

$$\begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix} \quad \begin{vmatrix} 1-x & -3 \\ 3 & -5-x \end{vmatrix}$$

$$= x^2+4x+4$$
$$= (x+2)^2$$

$$\lambda = -2 \quad \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A+2I)\eta = v$$

$$\left[\begin{array}{cc|c} 3 & -3 & 1 \\ 3 & -3 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{3} \\ 0 & 0 & 0 \end{array}\right]$$

$$\eta = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Take $\eta = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$

$$\boxed{\vec{x}(t) = C_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\left(te^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-2t}\begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}\right)}$$

Section 10.3

11) $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} d\theta + \frac{1}{2\pi}\int_{0}^{\pi} \theta\,d\theta$$

$$= \frac{1}{2\pi}(\pi) + \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) = \boxed{\frac{1}{2} + \frac{\pi}{4} = \hat{f}(0)}$$

For $n \neq 0$,

$u = \theta \quad dv = e^{-in\theta}d\theta$

$du = d\theta \quad v = \frac{e^{-in\theta}}{-in}$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{0} e^{-in\theta}\,d\theta + \frac{1}{2\pi}\int_{0}^{\pi} \theta e^{-in\theta}\,d\theta$$

$$= \frac{1}{2\pi}\left(\frac{e^{-in\theta}}{-in}\right)\Bigg|_{-\pi}^{0} + \frac{1}{2\pi}\left(\frac{\theta e^{-in\theta}}{-in}\Bigg|_{0}^{\pi} + \int_{0}^{\pi}\frac{e^{-in\theta}}{in}\,d\theta\right)$$

$$= \underbrace{\frac{1}{2\pi}\left(\frac{e^{-in\theta}}{-in}\right)\Bigg|_{-\pi}^{0}}_{1} + \frac{1}{2\pi}\left(\underbrace{\frac{\theta e^{-in\theta}}{-in}\Bigg|_{0}^{\pi}}_{2} + \underbrace{\frac{e^{-in\theta}}{-(in)^2}\Bigg|_{0}^{\pi}}_{3}\right)$$

$$1 = \frac{1}{2\pi}\left(-\frac{1}{in} + \frac{e^{in\pi}}{in}\right)$$

$$2 = \frac{1}{2\pi}\left(-\frac{1}{in}\right)\left(\pi e^{-in\pi}\right) \qquad 3 = \left(\frac{e^{-in\pi}}{-(in)^2} + \frac{1}{(in)^2}\right)\frac{1}{2\pi}$$

$$\hat{f}(n) = \frac{1}{2\pi}\left(-\frac{1}{in}\right)\left(1 - \underbrace{e^{in\pi}}_{=\cos(n\pi) + i\sin(n\pi) = (-1)^n} + \pi e^{-in\pi} + \frac{e^{-in\pi}}{in} - \frac{1}{in}\right)$$

$$\boxed{\hat{f}(n) = -\frac{1}{2\pi in}\left(1 - \frac{1}{in} + (\pi - 1)(-1)^n + \frac{(-1)^n}{in}\right) \text{ for } n \neq 0}$$

12) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$

$$\hat{f}(0) = \cancel{\frac{1}{2\pi}\int_{-\pi}^{0} 0 \, d\theta} + \frac{1}{2\pi}\int_0^{\pi} \theta^2 \, d\theta$$

$$= \frac{1}{2\pi}\left(\frac{\pi^3}{3}\right) = \boxed{\frac{\pi^2}{6} = \hat{f}(0)}$$

$n \neq 0$ $\quad \hat{f}(n) = \frac{1}{2\pi}\int_0^{\pi} \theta^2 e^{-in\theta} \, d\theta$

$u = \theta^2 \qquad dv = e^{-in\theta} d\theta$

$du = 2\theta d\theta \qquad v = \frac{e^{-in\theta}}{-in}$

$$= \frac{1}{2\pi}\left(\frac{\theta^2 e^{-in\theta}}{-in}\Big|_0^{\pi} + 2\int_0^{\pi} \frac{\theta e^{-in\theta}}{in} d\theta\right)$$

$u = 2\theta \qquad dv = \frac{e^{-in\theta}}{in} d\theta$

$du = 2d\theta \qquad v = \frac{e^{-in\theta}}{-(in)^2}$

$$= \frac{1}{2\pi}\left(-\frac{\theta^2 e^{-in\theta}}{in}\Big|_0^{\pi} + \frac{2\theta e^{-in\theta}}{-(in)^2}\Big|_0^{\pi} + 2\int_0^{\pi} \frac{e^{-in\theta}}{(in)^2} d\theta\right)$$

$$= \frac{1}{2\pi}\left(\underbrace{-\frac{\theta^2 e^{-in\theta}}{in}\Big|_0^{\pi}}_{①} + \underbrace{\frac{2\theta e^{-in\theta}}{-(in)^2}\Big|_0^{\pi}}_{②} + \underbrace{\frac{2e^{-in\theta}}{-(in)^3}\Big|_0^{\pi}}_{③}\right)$$

① $-\frac{\pi^2 e^{-in\pi}}{in} = -\frac{\pi^2(-1)^n}{in}$ ② $\frac{2\pi e^{-in\pi}}{-(in)^2} = \frac{-2\pi(-1)^n}{(in)^2}$

③ $\frac{2e^{-in\pi}}{-(in)^3} + \frac{2}{(in)^3} = -\frac{2(-1)^n}{(in)^3} + \frac{2}{(in)^3}$

So for $n \neq 0$ $\boxed{\hat{f}(n) = \frac{1}{2\pi}\left(-\frac{\pi^2(-1)^n}{in} - \frac{2\pi(-1)^n}{(in)^2} - \frac{2(-1)^n}{(in)^3} + \frac{2}{(in)^3}\right)}$

13) $f(x)=x^2, \quad -\pi < x < \pi$

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2 \, d\theta = \frac{\pi^2}{3}$$

For $n \neq 0$, $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2 e^{-in\theta} \, d\theta$

$$= \frac{1}{2\pi}\left(\underbrace{-\frac{\theta^2 e^{-in\theta}}{in}\Big|_{-\pi}^{\pi}}_{①} + \underbrace{\frac{2\theta e^{-in\theta}}{-(in)^2}\Big|_{-\pi}^{\pi}}_{②} + \underbrace{\frac{2e^{-in\theta}}{-(in)^3}\Big|_{-\pi}^{\pi}}_{③} \right)$$

① $$-\frac{\pi^2 e^{-in\pi}}{in} + \frac{(-\pi)^2 e^{in\pi}}{in}$$

$$= -\frac{\pi^2}{in}(-1)^n + \frac{\pi^2}{in}(-1)^n = 0.$$

② $$\frac{2\pi e^{-in\pi}}{-(in)^2} + \frac{2\pi e^{in\pi}}{-(in)^2} = \frac{4\pi(-1)^n}{n^2}$$

③ $$\frac{2e^{-in\pi}}{-(in)^3} - \frac{2e^{in\pi}}{-(in)^3} = \frac{2}{-(in)^3}\left(e^{-in\pi} - e^{in\pi}\right)$$

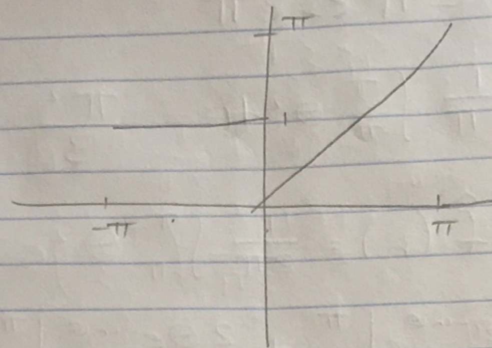
$$= \frac{2}{-(in)^3}\left((-1)^n - (-1)^n\right)$$

$$= 0.$$

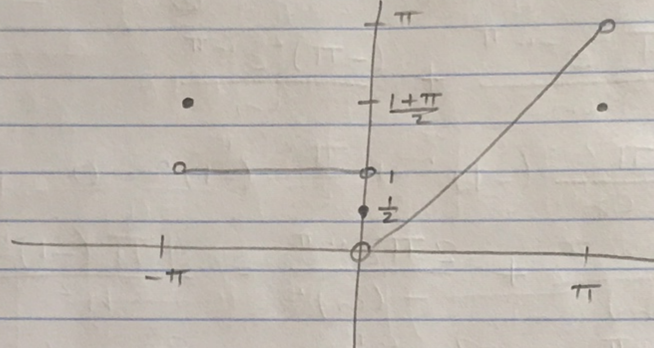
$$= \frac{1}{2\pi}\left(\frac{4\pi(-1)^n}{n^2}\right) = \frac{2(-1)^n}{n^2}$$

$$\boxed{f(\theta) \sim \frac{\pi^2}{3} + \sum_{\substack{n\neq 0; \\ n\in\mathbb{Z}}} \frac{2(-1)^n}{n^2} e^{in\theta}}$$

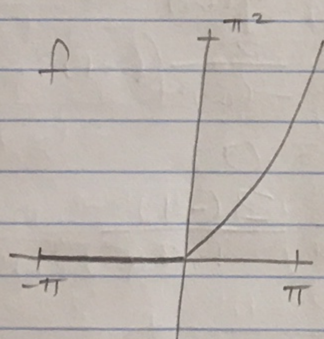
19) f



converges to



20) f



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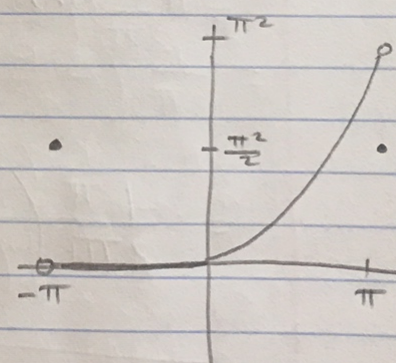
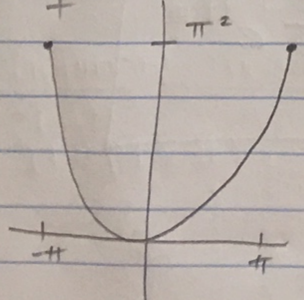


21) f



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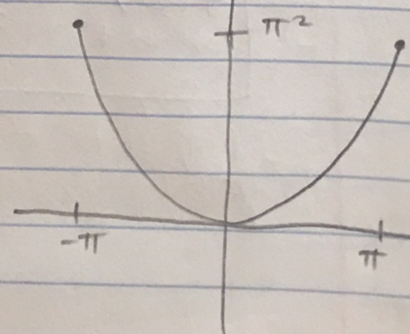


Additional Problems

1) $x''' - x'' - x' + x = 0$

$$\vec{x} = \begin{bmatrix} x \\ x' \\ x'' \end{bmatrix} \qquad \vec{x}\,' = \begin{bmatrix} x' \\ x'' \\ x''' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \\ -x + x' + x'' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ -1 & 1 & 1-x \end{vmatrix} = -x(-x + x^2 - 1) - 1(1)$$

$$= -x^3 + x^2 + x - 1$$

$$\begin{array}{r|rrrr} 1 & -1 & 1 & 1 & -1 \\ & & -1 & 0 & 1 \\ \hline & -1 & 0 & 1 & 0 \end{array}$$

$$= -(x-1)^2(x+1)$$

$\lambda = -1$ $\quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ $\quad v_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$\lambda = 1$ $\quad \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$(A - I)\eta = v_2$ $\quad \left[\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{array}\right]$

$$\vec{x} = \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix} = C_1 e^{-t}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$+ C_3\left(te^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + e^{t}\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}\right) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\eta = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So $x(t) = -C_1 e^{-t} + C_2 e^{t} + C_3 t e^{t} - 2C_3 e^{t}$

$x'(t) = C_1 e^{-t} + C_2 e^{t} + C_3 t e^{t} - C_3 e^{t}$

$x''(t) = -C_1 e^{-t} + C_2 e^{t} + C_3 t e^{t}$

$x(0) = 1 \Rightarrow -C_1 + C_2 + (-2C_3) = 1$

$x'(0) = -1 \Rightarrow C_1 + C_2 - C_3 = -1$

$x''(0) = 0 \Rightarrow -C_1 + C_2 = 0$

$C_3 = -2$

$C_1 = -\frac{3}{2} \quad C_2 = -\frac{3}{2}$

$x(t) = \frac{3}{2} e^{-t} - \frac{3}{2} e^{t} - 2te^{t} + 4e^{t}$

$= \boxed{\frac{3}{2} e^{-t} - 2te^{t} + \frac{5}{2} e^{t}}$

$x''' + x'' + x' + x = 0$

$\vec{x} = \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix} \qquad \vec{x}' = \begin{pmatrix} x' \\ x'' \\ x''' \end{pmatrix} = \begin{pmatrix} x' \\ x'' \\ -x - x' - x'' \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \vec{x}$

$\begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ -1 & -1 & -1-x \end{vmatrix} = -x(x^2+x+1) - 1(1)$

$= -x^3 - x^2 - x - 1 = -(x+1)(x^2+1)$

$$\begin{array}{r|rrrr} -1 & -1 & -1 & -1 & -1 \\ & & 1 & 0 & 1 \\ \hline & -1 & 0 & -1 & 0 \end{array}$$

$\lambda = -1 \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \qquad v_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$\lambda = i$ $\begin{bmatrix} -i & 1 & 0 \\ 0 & -i & 1 \\ -1 & -1 & -1-i \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & i & 0 \\ 0 & 1 & i \\ 1 & 1 & 1+i \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & i & 0 \\ 0 & 1 & i \\ 0 & 1-i & 1+i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - iR_2 \\ \\ R_3 + (i-1)R_2 \end{matrix}$

$v_2 = \begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix}$

$e^{it}\begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -\cos t \\ \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} -\sin t \\ -\cos t \\ \sin t \end{pmatrix}$

$\vec{x}(t) = \begin{bmatrix} x \\ x' \\ x'' \end{bmatrix} = C_1 e^{-t}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} -\cos t \\ \sin t \\ \cos t \end{pmatrix} + C_3\begin{pmatrix} -\sin t \\ -\cos t \\ \sin t \end{pmatrix}$

$x(t) = -C_1 e^{-t} - C_2\cos t - C_3 \sin t$

$x'(t) = C_1 e^{-t} + C_2 \sin t - C_3 \cos t$

$x''(t) = -C_1 e^{-t} + C_2 \cos t + C_3 \sin t$

$x(0) = 1 \quad x'(0) = -1 \quad x''(0) = 0$

$-C_1 - C_2 = 1$

$C_1 - C_3 = -1$

$-C_1 + C_2 = 0$

$C_1 = -\frac{1}{2} \quad C_2 = -\frac{1}{2}$

$C_3 = \frac{1}{2}$

$$\boxed{x(t) = \frac{1}{2}e^{-t} + \frac{1}{2}\cos t - \frac{1}{2}\sin t}$$

2) $\widehat{f'}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{-ik\theta}\,d\theta$

$u = e^{-ik\theta}$ $\quad dv = f'(\theta)d\theta$

$du = (-ik)e^{-ik\theta}d\theta$ $\quad v = f(\theta)$

$$= \frac{1}{2\pi}\left(\underbrace{\cancel{f(\theta)e^{-ik\theta}\Big|_{-\pi}^{\pi}}}_{\substack{=0 \text{ since } f(-\pi)=f(\pi) \\ \text{and } e^{ik\pi}=e^{-ik\pi}=(-1)^k.}} + ik\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\,d\theta \right)$$

$$= ik\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\,d\theta\right)$$

$$= ik\hat{f}(k).$$

So $\widehat{f'}(k) = ik\hat{f}(k)$

and by iterating, $\widehat{f^{(N)}}(k) = (ik)^N\hat{f}(k)$.

$$|k^N\hat{f}(k)| = |(ik)^N\hat{f}(k)| = |\widehat{f^{(N)}}(k)|$$

$$= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(N)}(\theta)e^{-ik\theta}\,d\theta\right|$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f^{(N)}(\theta)||e^{-ik\theta}|\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f^{(N)}(\theta)|\,d\theta$$

On $[-\pi,\pi]$, $|f^{(N)}(\theta)| \le M_N$ for some constant M_N by the Extreme Value Theorem

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} M_N\,d\theta = M_N$$

So $|k|^N|\hat{f}(k)| \le M_N$

$\Rightarrow |\hat{f}(k)| \le M_N |k|^N.$

3) $f(\theta) = |\theta| = \begin{cases} \theta, & 0 < \theta < \pi \\ -\theta, & -\pi < \theta < 0 \end{cases}$

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{0} -\theta\,d\theta + \frac{1}{2\pi}\int_{0}^{\pi} \theta\,d\theta$$
$$= \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) + \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

For $n \neq 0$,

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{0} -\theta e^{-in\theta}\,d\theta + \frac{1}{2\pi}\int_{0}^{\pi} \theta e^{-in\theta}\,d\theta$$

(See Answer Key to Practice Exam 4, Problem 6 for full computation)

The Fourier series of f converges to f since f is continuous and f' is piecewise continuous.

So $$|\theta| = \sum_{\substack{n\neq 0,\\ n\in\mathbb{Z}}} \frac{\cos(n\pi)-1}{\pi n^2} e^{in\theta} + \frac{\pi}{2}$$

Set $\theta = 0$. $$\frac{\pi}{2} + \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{(-1)^n - 1}{\pi n^2} = 0.$$

$|\theta|\big|_{\theta=0} = 0.$

(0 if n is even)

$$\frac{\pi}{2} + \sum_{\substack{n \text{ odd},\\ n\neq 0}} \frac{-2}{\pi n^2} = 0.$$

n pos. and neg. $\frac{2}{\pi n^2} = \frac{2}{\pi(-n)^2}$

only n pos.

$$\Rightarrow \frac{\pi}{2} + \sum_{\substack{n>0\\ n \text{ odd}}} \frac{-4}{\pi n^2} = 0.$$

So $\frac{4}{\pi} \sum_{\substack{n \text{ odd} \\ n \geq 1}} \frac{1}{n^2} = \frac{\pi}{2}$

$$\sum_{\substack{n \text{ odd} \\ n \geq 1}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \underbrace{\sum_{\substack{n \text{ odd,} \\ n \geq 1}} \frac{1}{n^2}}_{\frac{\pi^2}{8}} + \underbrace{\sum_{\substack{n \text{ even} \\ n \geq 1}} \frac{1}{n^2}}_{= \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}}$$

So $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

4) Show $c_1 e^{a_1 t} + c_2 e^{a_2 t} + \dots + c_n e^{a_n t} = 0$
only has trivial soln.

Differentiate $n-1$ times.

$$e^{a_1 t} c_1 + e^{a_2 t} c_2 + \dots + e^{a_n t} c_n = 0$$
$$a_1 e^{a_1 t} c_1 + a_2 e^{a_2 t} c_2 + \dots + a_n e^{a_n t} c_n = 0$$
$$\vdots$$
$$a_1^{n-1} e^{a_1 t} c_1 + a_2^{n-1} e^{a_2 t} c_2 + \dots + a_n^{n-1} e^{a_n t} c_n = 0$$

Set $t = 0$:

$$c_1 + c_2 + \dots + c_n = 0$$
$$a_1 c_1 + a_2 c_2 + \dots + a_n c_n = 0$$
$$\vdots$$
$$a_1^{n-1} c_1 + a_2^{n-1} c_2 + \dots + a_n^{n-1} c_n = 0$$

But $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}$

is invertible since a_i are all distinct
(it's the inverse of the Vandermonde matrix).

So we only have the trivial soln c_i all 0.

5) Set $n=2$ for simplicity.

Since T is diagonalizable, there exists a basis $\mathcal{B}=\{v_1, v_2\}$ for $\mathbb{R}^2$ consisting of eigenvectors of T, and

$$T(v_1)=\lambda_1 v_1 \quad T(v_2)=\lambda_2 v_2$$

so $[T]_{\mathcal{B}\to\mathcal{B}}$ is diagonal. But $\mathcal{B}$ is not necessarily orthonormal.

So construct a new orthonormal basis from $\mathcal{B}$ by Gram-Schmidt. Let

$$w_1 = \frac{v_1}{|v_1|} \qquad w_2 = \frac{v_2 - \langle v_2, w_1\rangle w_1}{|v_2 - \langle v_2, w_1\rangle w_1|}$$

$\mathcal{C} = \{w_1, w_2\}$ is now an orthonormal basis for $\mathbb{R}^2$.

$$T(w_1) = \frac{1}{|v_1|} T(v_1) = \frac{\lambda_1 v_1}{|v_1|} = \lambda_1 w_1$$

and $T(w_2) = c_1 w_1 + c_2 w_2$ for some c_1, c_2.

$$\text{So } [T]_{\mathcal{C}\to\mathcal{C}} = \begin{bmatrix} \lambda_1 & c_1 \\ 0 & c_2 \end{bmatrix}$$

is upper triangular where $\mathcal{C}$ is an orthonormal basis for $\mathbb{R}^2$.

Since the matrix is symmetric, all eigenvalues are real.

6) Use strong induction and Sylvester's criterion, to show A is positive definite.

(induction on size of A)

Base Case: $(n=1)$ $|2| = 2$

$(n=2)$ $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

and $3 > 2$

Denote by A_n the $n \times n$ matrix, where we induct on n.

Our inductive hypothesis is that:

For all $n \leq k$, $\det(A_n) > 0$ and $\det(A_k) > \det(A_{k-1}) > \cdots > \det(A_2) > \det(A_1) > 0$

Check inductive step: We just need to show that $\det(A_{k+1}) > \underbrace{\det(A_k) > 0}_{\text{inductive step}}$.

But expanding along the last column,

$$\det(A_{k+1}) = 2\det(A_k) + 1\det\begin{pmatrix} A_{k-1} & \begin{matrix} 0 \\ \vdots \\ 0 \\ -1 \end{matrix} \\ 0 \cdots 0 & -1 \end{pmatrix}$$

$$= 2\det(A_k) + 1\det(A_{k-1}) \cdot (-1)$$

$$= 2\det(A_k) - \det(A_{k-1}) > \det(A_k)$$

since $\det(A_k) > \det(A_{k-1})$ by the inductive assumption.

So by induction, $\det(A_n)$ is an increasing positive sequence. So by Sylvester's Criterion, A is positive definite and hence has real positive eigenvalues.

7) (a) $$(\widehat{f*g})(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(\theta)e^{-ik\theta}\,d\theta$$

$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\eta)g(\theta-\eta)\,d\eta\, e^{-ik\theta}\,d\theta$$

$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\eta)g(\theta-\eta)e^{-ik\eta}e^{-ik(\theta-\eta)}\,d\eta\,d\theta$$

(Fubini)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta)e^{-ik\eta}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta-\eta)e^{-ik(\theta-\eta)}\,d\theta\right)d\eta$$

$\theta-\eta \to \theta$

change of var.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta)e^{-ik\eta}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-ik\theta}\,d\theta\right)d\eta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta)e^{-ik\eta}\,\hat{g}(k)\,d\eta$$

$$= \hat{g}(k)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta)e^{-ik\eta}\,d\eta\right)$$

$$= \hat{g}(k)\hat{f}(k) = \hat{f}(k)\hat{g}(k)$$

(b) Note that $\displaystyle\sum_{k\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}}\hat{f}(n)\hat{g}(k-n)\right)e^{-ik\theta}$

Let $f, g \in C^\infty(S^1)$

$$= \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \hat{f}(n)\hat{g}(k-n)e^{-i(k-n)\theta}e^{-in\theta}$$

$$= \sum_{n\in\mathbb{Z}}\left(\hat{f}(n)e^{-in\theta}\left(\sum_{k\in\mathbb{Z}}\hat{g}(k-n)e^{-i(k-n)\theta}\right)\right)$$

$k-n \to k$

$$= \sum_{n\in\mathbb{Z}}\left(\hat{f}(n)e^{-in\theta}\left(\sum_{k\in\mathbb{Z}}\hat{g}(k)e^{-ik\theta}\right)\right)$$

$$= \sum_{n\in\mathbb{Z}}\hat{f}(n)e^{-in\theta}g(\theta) = f(\theta)g(\theta).$$

So $\displaystyle\widehat{fg} = \sum_{n\in\mathbb{Z}}\hat{f}(n)\hat{g}(k-n)$

8) (a) $(\partial_t + \partial_x)(\partial_t - \partial_x)u$

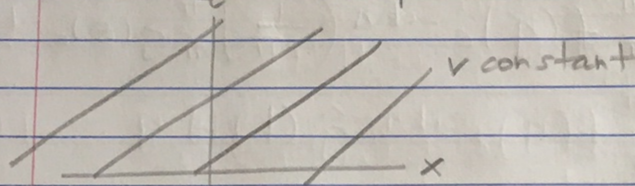
$$= (\partial_t + \partial_x)(u_t - u_x)$$

$$= u_{tt} + \cancel{u_{xt} - u_{tx}} - u_{xx}$$

$$= u_{tt} - u_{xx} = \partial_t^2 u - \partial_x^2 u$$

(b) $\partial_t + \partial_x$ is a $(1,1)$ directional derivative.

So $(\partial_t + \partial_x)v = 0$ has the solution, where v is constant along every line of slope 1.



So if $(\partial_t + \partial_x)[(\partial_t - \partial_x)u] = 0$,

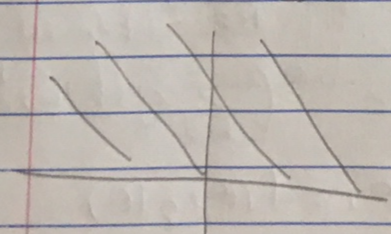
then $(\partial_t - \partial_x)u(x,t) = (\partial_t - \partial_x)u(x-t)$

$$= g(x-t) - f_x(x-t)$$

(initial conditions).

x, t

Now, $\partial_t - \partial_x$ is a $(-1,1)$ directional derivative. The unit vector is $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Use FTC along these lines.



$$\gamma(s) = \left(x+t-\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

$$u(\gamma(s_0)) = \int_0^{s_0} \frac{1}{\sqrt{2}}\left(g\left(\left(x+t-\frac{s}{\sqrt{2}}\right)-\frac{s}{\sqrt{2}}\right) - f_x\left(\left(x+t-\frac{s}{\sqrt{2}}\right)-\frac{s}{\sqrt{2}}\right)\right)ds$$

$$+ f(x+t)$$ ← initial cond.

$$\text{So } u(\gamma(s_0)) = \frac{1}{\sqrt{2}} \int_0^{s_0} g(x+t-\sqrt{2}s) - f_x(x+t-\sqrt{2}s)\,ds + f(x+t)$$

$$\gamma(s) = \left(x+t-\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

so $\gamma(s_0) = (x,t)$ when $s_0 = \sqrt{2}t$.

$$\text{So } u(x,t) = u(\gamma(\sqrt{2}t))$$

$$= \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}t} g(x+t-\sqrt{2}s) - f_x(x+t-\sqrt{2}s)\,ds + f(x+t)$$

$u = \sqrt{2}s \qquad ds = \frac{1}{\sqrt{2}}du$

$$= \frac{1}{2} \int_0^{2t} g(x+t-u) - f_x(x+t-u)\,du + f(x+t)$$

$$= \frac{1}{2} \int_0^{2t} g(x+t-u) - \frac{1}{2}\int_0^{2t} f_x(x+t-u)\,du + f(x+t)$$

$$= \frac{1}{2} \int_{-t}^{t} g(x+s)\,ds - \frac{1}{2}\left(-f(x+t-u)\Big|_{u=0}^{u=2t}\right) + f(x+t)$$

$$= \frac{1}{2} \int_{-t}^{t} g(x+s)\,ds + \frac{1}{2}f(x-t) - \frac{1}{2}f(x+t) + f(x+t)$$

$$\text{So } \boxed{u(x,t) = \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{-t}^{t} g(x+s)\,ds}$$