Problem Set 10 Solutions

Section 4.2

6) $y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0 \qquad (r-3)(r-2) = 0$

$r = 2, 3$

$$\boxed{y = C_1 e^{2x} + C_2 e^{3x}}$$

10) $y'' - y' - 11y = 0$

$r^2 - r - 11 = 0 \qquad r = \frac{1 \pm \sqrt{1+44}}{2} = \frac{1 \pm \sqrt{45}}{2}$

$= \frac{1}{2} \pm \frac{3\sqrt{5}}{2}$

$$\boxed{y = C_1 e^{\left(\frac{1}{2} + \frac{3\sqrt{5}}{2}\right)x} + C_2 e^{\left(\frac{1}{2} - \frac{3\sqrt{5}}{2}\right)x}}$$

18) $z'' - 2z' - 2z = 0 \qquad z(0) = 0, \; z'(0) = 3$

$r^2 - 2r - 2 = 0$

$r = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$

$z = C_1 e^{(1+\sqrt{3})x} + C_2 e^{(1-\sqrt{3})x}$

$z(0) = 0 \qquad C_1 + C_2 = 0$

$z'(0) = 3$

$z' = C_1(1+\sqrt{3})e^{(1+\sqrt{3})x} + C_2(1-\sqrt{3})e^{(1-\sqrt{3})x}$

$C_1(1+\sqrt{3}) + C_2(1-\sqrt{3}) = 3$

$C_1 = \frac{\sqrt{3}}{2} \quad C_2 = -\frac{\sqrt{3}}{2}$

$$\boxed{z = \frac{\sqrt{3}}{2} e^{(1+\sqrt{3})x} - \frac{\sqrt{3}}{2} e^{(1-\sqrt{3})x}}$$

Section 4.3

10) $y'' + 4y' + 8y = 0$

$r^2 + 4r + 8 = 0$

$r = \dfrac{-4 \pm \sqrt{-16}}{2} = -2 \pm 2i$

$\boxed{y = C_1 e^{-2t}\cos(2t) + C_2 e^{-2t}\sin(2t)}$

16) $y'' + 10y' + 41y = 0$

$r^2 + 10r + 41 = 0$

$r = \dfrac{-10 \pm \sqrt{-64}}{2} = -5 \pm 4i$

$\boxed{y = C_1 e^{-5t}\cos(4t) + C_2 e^{-5t}\sin(4t)}$

22) $y'' + 2y' + 17y = 0, \; y(0) = 1, \; y'(0) = -1$

$r^2 + 2r + 17 = 0 \quad r = \dfrac{-2 \pm \sqrt{-64}}{2} = -1 \pm 4i$

$y = C_1 e^{-t}\cos(4t) + C_2 e^{-t}\sin(4t)$

$1 = C_1 \qquad y' = -C_1 e^{-t}\cos(4t) - 4C_1 e^{-t}\sin(4t)$
$\qquad - C_2 e^{-t}\sin(4t) + 4C_2 e^{-t}\cos(4t)$

$-C_1 + 4C_2 = -1 \qquad C_2 = 0.$

$\boxed{y = e^{-t}\cos(4t)}$

Section 4.4

18) $y'' - 2y' + y = 8e^t$ $\qquad y_h = C_1 e^t + C_2 t e^t$

$y_p = \underbrace{Ae^t}_{\text{in } y_h}$ Basic guess ✗

⇓

$y_p = Ate^t$ But still in y_h ✗

⇓

$y_p = At^2e^t$ ✓

$y_p' = At^2e^t + 2Ate^t$

$y_p'' = At^2e^t + 4Ate^t + 2Ae^t$

$\cancel{At^2e^t} + \cancel{4Ate^t} + 2Ae^t - \cancel{2At^2e^t} - \cancel{4Ate^t} + \cancel{At^2e^t} = 8e^t$

$2Ae^t = 8e^t \quad A = 4$

$$\boxed{y_p = 4t^2e^t}$$

20) $y'' + 4y = 16t\sin(2t)$

$r^2 + 4 = 0 \quad r = \pm 2i$

$y_h = C_1\sin(2t) + C_2\cos(2t)$

y_p Basic Guess $(At + B)(C\cos(2t) + D\sin(2t))$

$\rightarrow$ Consolidate constants

$At\cos(2t) + Bt\sin(2t) + \underbrace{C\cos(2t) + D\sin(2t)}_{\text{in } y_h \ \times}$

$\Downarrow$ Multiply by x

$y_p = At^2\cos(2t) + Bt^2\sin(2t) + Ct\cos(2t) + Dt\sin(2t)$

$= (At^2 + Ct)\cos(2t) + (Bt^2 + Dt)\sin(2t)$ ✓

$y_p' = -2(At^2 + Ct)\sin(2t)$

$+ (2At + C)\cos(2t) + 2(Bt^2 + Dt)\cos(2t)$

$+ (2Bt + D)\sin(2t)$

$= (-2At^2 + (2B - 2C)t + D))\sin(2t)$

$+ (2Bt^2 + (2A + 2D)t + C)\cos(2t)$

$y_p'' = 2(-2At^2 + (2B - 2C)t + D)\cos(2t)$

$+ (-4At + (2B - 2C))\sin(2t)$

$+ (-2)(2Bt^2 + (2A + 2D)t + C)\sin(2t)$

$+ (4Bt + (2A + 2D))\cos(2t)$

$= (-4At^2 + (8B - 4C)t + (2A + 4D))\cos(2t)$

$+ (-4Bt^2 + (-8A - 4D)t + (2B - 4C))\sin(2t)$

$y_p'' + 4y = (8Bt + (2A + 4D))\cos(2t)$

$+ (-8At + (2B - 4C))\sin(2t)$

$= 16t\sin(2t)$

$8B = 0 \quad 2A + 4D = 0 \quad -8A = 16 \quad 2B - 4C = 0$

$A = -2 \quad B = 0 \quad C = 0 \quad D = 1$

$$\boxed{y_p = -2t^2\cos(2t) + t\sin(2t)}$$

Section 4.5

18) $y'' - y = -11t + 1$

$r^2 - 1 = 0 \quad r = \pm 1$

$y_h = C_1 e^t + C_2 e^{-t}$

$y_p = At + B \quad y_p' = A \quad y_p'' = 0$

$0 - (At + B) = -11t + 1$

$A = 11, B = -1$

$$\boxed{y = 11t - 1 + C_1 e^t + C_2 e^{-t}}$$

22) $y'' + 6y' + 10y = 10x^4 + 24x^3 + 2x^2 - 12x + 18$

$r^2 + 6r + 10 = 0 \quad r = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i$

$y_h = C_1 e^{-3x} \cos x + C_2 e^{-3x} \sin x$

$y_p = Ax^4 + Bx^3 + Cx^2 + Dx + E$

$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D$

$y_p'' = 12Ax^2 + 6Bx + 2C$

$(12Ax^2 + 6Bx + 2C) + 6(4Ax^3 + 3Bx^2 + 2Cx + D)$

$+ 10(Ax^4 + Bx^3 + Cx^2 + Dx + E)$

$= 10x^4 + 24x^3 + 2x^2 - 12x + 18$

$10A = 10 \Rightarrow A = 1 \quad 10B + 24A = 24 \Rightarrow B = 0$

$12A + 18B + 10C = 2 \Rightarrow C = -1$

$6B + 12C + 10D = -12 \Rightarrow D = 0 \quad 2C + 6D + 10E = 18 \Rightarrow E = 2$

Section 4.6

2) $y'' + y = \sec t$

$y_h = C_1 \cos t + C_2 \sin t$

$y_1(t) = \cos t \quad y_2(t) = \sin t$

$y_1' = -\sin t \quad y_2' = \cos t$

$$v_1(t) = \int \frac{-\sec t \sin t}{\cos^2 t + \sin^2 t}\,dt$$

$$= \int -\tan t \, dt = \ln|\cos t| + C$$

$$v_2(t) = \int \frac{\sec t \cos t}{\cos^2 t + \sin^2 t}\,dt = \int 1\,dt = t + C$$

$$y_p = \cos t(\ln|\cos t|) + (\sin t)\,t$$

$$\boxed{y = \cos t(\ln|\cos t|) + t\sin t + C_1\cos t + C_2 \sin t}$$

6) $y'' + 9y = \sec^2(3t)$

$y_1(t) = \cos(3t) \quad y_2(t) = \sin(3t)$

$y_1'(t) = -3\sin(3t) \quad y_2'(t) = 3\cos(3t)$

$$W(t) = 3\cos^2(3t) - (-3\sin^2(3t)) = 3$$

$$v_1(t) = \int \frac{-\sec^2(3t)\sin(3t)}{3}\,dt = \int \frac{-\sin(3t)}{3\cos^2(3t)}\,dt$$

Let $u = \cos(3t)$

$du = -3\sin(3t)\,dt$

$$= \int \frac{1}{9u^2}\,du = \frac{-1}{9u} + C = \frac{-1}{9\cos(3t)} + C$$

$$v_2(t) = \int \frac{\sec^2(3t)\cos(3t)}{3}\,dt = \int \frac{1}{3}\sec(3t)\,dt$$

$$= \frac{1}{9}\ln|\sec(3t) + \tan(3t)| + C$$

$$\boxed{y = -\frac{1}{9} + \frac{1}{9}\sin(3t)\ln|\sec(3t) + \tan(3t)| + C_1\cos(3t) + C_2\sin(3t)}$$

8) $y'' + 4y = \csc^2(2t)$ $\quad y_1 = \cos(2t)$ $\quad y_2 = \sin(2t)$

$\quad y_1' = -2\sin(2t)$ $\quad y_2' = 2\cos(2t)$

$W(t) = y_1 y_2' - y_2 y_1' = 2$

$$v_1(t) = \int \frac{-\csc^2(2t)\sin(2t)}{2}\,dt$$

$$= \int -\frac{\csc(2t)}{2}\,dt$$

$$= \frac{1}{4}\ln|\csc(2t) + \cot(2t)| + C$$

$$v_2(t) = \int \frac{\csc^2(2t)\cos(2t)}{2}\,dt$$

$$= \int \frac{\cos(2t)}{2\sin^2(2t)}\,dt = -\frac{1}{4\sin(2t)} + C$$

$$\boxed{y = \frac{1}{4}\cos(2t)\ln|\csc(2t) + \cot(2t)| - \frac{1}{4} + C_1\cos(2t) + C_2\sin(2t)}$$

18) $y'' - 6y' + 9y = t^{-3}e^{3t}$

$y_1 = e^{3t}$ $\quad$ $y_2 = te^{3t}$

$y_1' = 3e^{3t}$ $\quad$ $y_2' = 3te^{3t} + e^{3t}$

$W = \cancel{3te^{6t}} + e^{6t} - \cancel{3te^{6t}} = e^{6t}$

$v_1 = \int \frac{-t^{-3}\cancel{e^{3t}}\, t\cancel{e^{3t}}}{\cancel{e^{6t}}}\, dt = \int -\frac{1}{t^2}\, dt = \frac{1}{t} + C$

$v_2 = \int \frac{t^{-3}e^{3t}e^{3t}}{e^{6t}}\, dt = \int \frac{1}{t^3}\, dt = -\frac{1}{2t^2} + C$

$y = \frac{e^{3t}}{t} - \frac{e^{3t}}{2t} + C_1e^{3t} + C_2te^{3t}$

$$\boxed{y = \frac{e^{3t}}{2t} + C_1e^{3t} + C_2te^{3t}}$$

Problem 1

$y'' + by' + 4y = 0$ $\quad$ $r^2 + br + 4 = 0$

$b^2 - 16$ discriminant

$b<-4, b>4 \rightarrow$ Case 1: Two real roots distinct when $b > 4$.
$\lim_{t\to\infty} y = 0$ always, when both roots $C_1e^{r_1t} + C_2e^{r_2t}$ are negative. $r = \frac{-b \pm \sqrt{b^2-16}}{2} < 0$ when $b > 4$

$b=-4, 4 \rightarrow$ Case 2: Repeated root when $b = 4$. $C_1e^{r_1t} + C_2te^{r_1t}$ when repeated root is negative gives $\lim_{t\to\infty} y = 0$. This happens only for $b = 4$.

$-4<b<4 \rightarrow$ Case 3: Two complex roots. $C_1e^{\alpha t}\cos\beta t + C_2e^{\alpha t}\sin\beta t$ $\rightarrow 0$ always when $t\to\infty$ exactly when $Re(\alpha \pm \beta i) = \alpha < 0$.
$Re\left(\frac{-b \pm \sqrt{b^2-16}}{2}\right) = -\frac{b}{2}$ so $b>0$ (and <4). Combining all three cases, $\boxed{b>0}$