A Visual Introduction to Partial Differential Equations

Math N54: Linear Algebra and Differential Equations

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Table of Contents

Table of Contents

What is a partial differential equation?

Over the course of linear algebra and differential equations, we have studied many different types of functions. We have thought of functions abstractly as "machines" that take in a certain input and give out a certain output.

Examples of functions in linear algebra

- Real-valued functions $f : \mathbb{R} \to \mathbb{R}$
- The matrix multiplication map given by an m by n matrix, $T : \mathbb{R}^n \to \mathbb{R}^m$
- A linear transformation between two abstract vector spaces, $T : V \to W$
- A linear functional $\ell : V \to \mathbb{R}$
- Complex-valued functions for Fourier series, $f : [-\pi, \pi) \to \mathbb{C}$

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- The profit function given certain variables, $P : \mathbb{R}^n \to \mathbb{R}$
- Chemical concentration as a function of (x, y) position, $F : \mathbb{R}^2 \to \mathbb{R}$

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Just as for ordinary differential equations, many of these quantities satisfy differential equations. However, since these are multi-dimensional functions now, we have to understand what it means to take the derivative of a multi-dimensional function.

Visualizing Multi-Dimensional Functions

First, let's start by considering functions of the form $f : \mathbb{R}^2 \to \mathbb{R}$. For example, let us consider

$$f(x, y) = x^2 + y^2$$

How can we visualize this?

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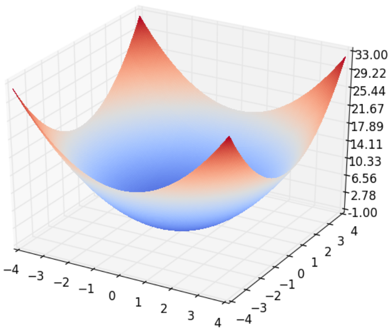

How to understand a multi-dimensional derivative

We can understand derivatives as slopes of tangent lines, but there are many different tangent lines here!

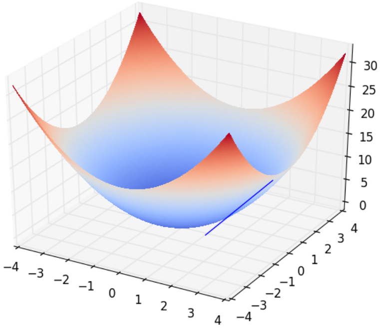

Figure: The graph of $f(x, y) = x^2 + y^2$ with partial x derivative tangent line

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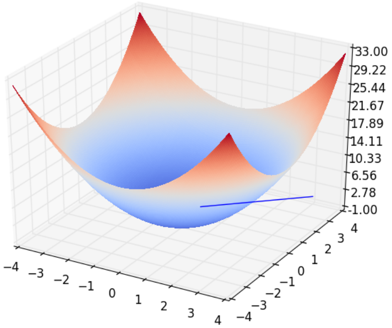

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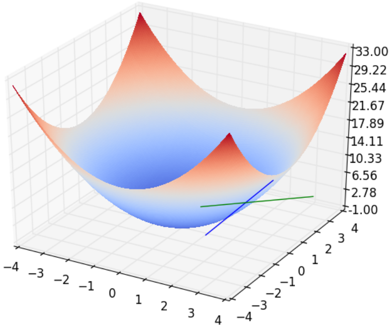

Figure: The graph of $f(x, y) = x^2 + y^2$ with partial x (blue), y (green) derivative tangent lines

Partial derivatives

So there are many different tangent lines to a function $f(x, y)$ at any given point!

Partial derivatives

So there are many different tangent lines to a function $f(x, y)$ at any given point! Since the x and y directions span $\mathbb{R}^2$, we are particularly interested in the tangent line where y is kept constant and the tangent line where x is kept constant. These are defined as **partial derivatives** of a function, and are denoted by $\partial_x f$ and $\partial_y f$ respectively.

Partial derivatives

To take a partial derivative, differentiate with respect to the desired variable, and treat all other variables as “constants”.

Examples

- $\frac{\partial}{\partial x}(e^{2x}y^2) = 2e^{2x}y^2$
- $\frac{\partial}{\partial y}(e^{2x}y^2) = 2e^{2x}y$
- $\frac{\partial}{\partial x}(\sin(xy^2)) = y^2\cos(xy^2)$
- $\frac{\partial}{\partial y}(\sin(xy^2)) = 2xy\cos(xy^2)$

What is a partial differential equation?

A **partial differential equation** is a equation involving a condition on the partial derivatives of a function.

Important Partial Differential Equations

- Laplace's equation (chemical equilibrium, wave physics) $\Delta f = 0$
- Laplace eigenvalue equation (vibration and resonance, electron orbitals) $-\Delta f = \lambda f$
- Heat equation (chemistry, thermodynamics) $\partial_t f - \Delta f = 0$
- Wave equation (physics) $\partial_t^2 f - \Delta f = 0$
- Burger's equation (gas dynamics) $\partial_t f + f\partial_x f = 0$
- And more! (Navier-Stokes equation for fluid flow, civil engineering, mechanical engineering; Korteweg-deVries equation for water waves; Einstein's equation for cosmology and general relativity)

Table of Contents

Definitions

Let's first start with some basic definitions.

Definition

The *Laplacian* is a second order partial differential operator, given in $\mathbb{R}^n$ by

$$\Delta = \frac{\partial}{\partial x_1}^2 + \frac{\partial}{\partial x_2}^2 + ... + \frac{\partial}{\partial x_n}^2$$

We will consider the case of $n = 2$, since we are considering functions $f(x, y)$. In this case,

$$\Delta f = \partial_x^2 f + \partial_y^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

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Definition

A harmonic function on $\mathbb{R}^n$ is a function f such that $\Delta f = 0$. So harmonic solutions are solutions to Laplace's partial differential equation $\Delta f = 0$.

Physical interpretation of Laplace's equation

$$\Delta f = 0 \quad \text{(Laplace's equation)}$$

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Laplace's equation describes equilibrium states. It can represent for example the equilibrium state of a chemical, where there are varying concentrations of the chemical over all of $\mathbb{R}^2$.

Physical interpretation of Laplace's equation

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Laplace's equation describes equilibrium states. It can represent for example the equilibrium state of a chemical, where there are varying concentrations of the chemical over all of $\mathbb{R}^2$.

To see this mathematically, use the divergence theorem. On any smooth region U,

$$\int_U \Delta u dx = \int_U \text{div}(\nabla u) dx = \int_{\partial U} \nabla u \cdot \mathbf{n} dS$$

so if $\Delta u = 0$, then

$$\int_{\partial U} \nabla u \cdot \mathbf{n} dS = 0$$

which means there is no net flux (no net diffusion) in or out of any region.

Examples of harmonic functions

Examples

Here are examples of harmonic functions:

- $f(x, y) = c$ where c is a constant
- $f(x, y) = ax + by + c$, any linear function of x and y
- $f(x, y) = x^2 - y^2$
- $f(x, y) = e^x \cos(y)$
- $f(x, y) = x^3 - 3xy^2$

Examples of harmonic functions

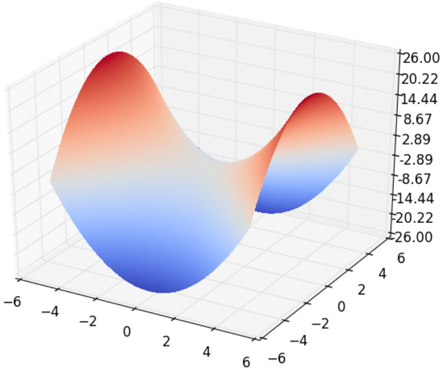
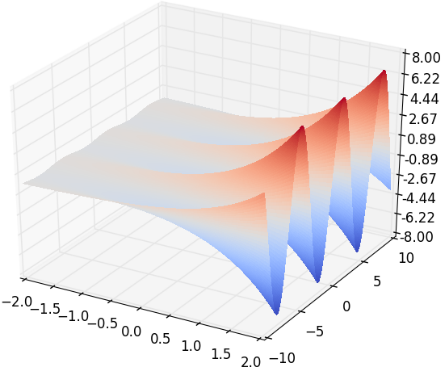


Figure: The harmonic function $f(x, y) = x^2 - y^2$

Examples of harmonic functions



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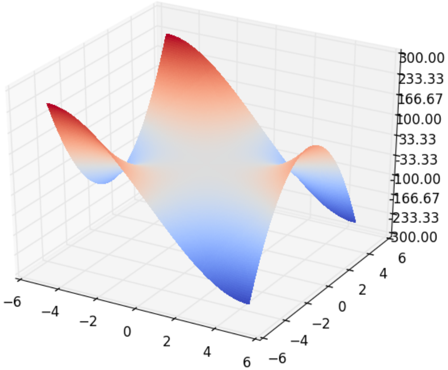

Figure: The harmonic function $f(x, y) = x^3 - 3xy^2$

Mean value principle

Mean value principle

Let f be a harmonic function on $\mathbb{R}^n$. Then f satisfies the mean value property:

$$f(x) = \fint_{\partial B(x,r)} f(y)dy$$

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If we remember the physical interpretation of f as an equilibrium, this makes sense.

Maximum and minimum principle

We can use the mean value theorem to deduce the following.

Maximum and minimum principle

Let f be a harmonic function on $\mathbb{R}^n$. The maximum of f on any ball cannot occur on the boundary unless f is constant. The same statement holds for minima.

$$\max_{y \in B(x,r)} f(y) = \max_{y \in \partial B(x,r)} f(y)$$

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This makes sense physically. It is not possible for a nonconstant equilibrium state to have a point at which a chemical concentration is largest or smallest, else diffusion would cause the state to not be equilibrium.

Regularity

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Regularity of solutions to Laplace's equation

Every solution to the Laplace's equation has infinitely many derivatives, and has a convergent Taylor series in a neighborhood around every point.

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Every solution to the Laplace's equation has infinitely many derivatives, and has a convergent Taylor series in a neighborhood around every point.

This fact is called **regularity**. The actual solutions to the equation have much more smoothness than is required by the actual equation itself.

Table of Contents

Definitions

Dirichlet eigenvalues of the Laplacian Δ

Let Ω be any open set in $\mathbb{R}^n$. Dirichlet eigenvalue equation for the Laplacian Δ is

$$-\Delta f = \lambda f \text{ on } \Omega$$

$$f|_{\partial\Omega} = 0$$

where λ is the associated eigenvalue for the eigenfunction f. So we are looking for eigenfunctions that are zero on the boundary of the set we are considering.

You can think of this as the resonant frequencies emitted by a drum that is struck whose surface has the shape given by the set Ω in $\mathbb{R}^2$, where the eigenvalues λ are the resonant pitches (frequencies).

Why is this equation important?

Decomposition into resonant frequencies is related to Fourier series and spectral decomposition.

Example

In one dimension (so on $\mathbb{R}$), the Laplacian is just a second derivative. If we consider the set $\Omega = (0, 1)$, note that $\sin(n\pi x)$ are Dirichlet eigenfunctions of $-\Delta$ on Ω since

$$-\Delta(\sin(n\pi x)) = (n^2\pi^2)\sin(n\pi x)$$

$$\sin(n\pi x) = 0 \text{ at } x = 0, 1$$

The eigenvector $f_n = \sin(\pi n x)$ corresponds to the eigenvalue $n^2\pi^2$, $n \geq 1$.

Dirichlet eigenfunctions for $-\Delta$ on $\Omega = (0, 1)$ in $\mathbb{R}$

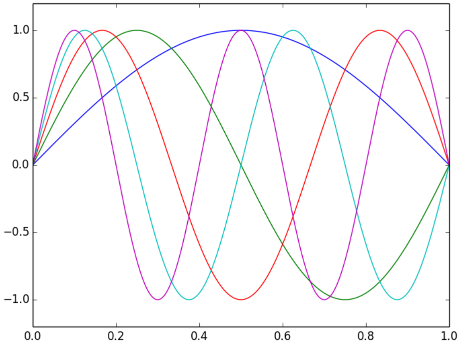

Figure: The first five Dirichlet eigenvalues of $-\Delta$ on $(0, 1)$ in $\mathbb{R}$.

These represent the vibrations of a string that is held still at $x = 0, 1$.

Dirichlet eigenfunctions for $-\Delta$ on a square in $\mathbb{R}^2$

Now consider Ω to be the square $(0,1) \times (0,1)$ in $\mathbb{R}^2$. Let's consider the Dirichlet eigenfunctions here.

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Using a technique in PDEs known as separation of variables, where we guess solutions to the PDE of the form $f(x,y) = g(x)h(y)$, we can see that the Dirichlet eigenfunctions for $-\Delta$ here are

$$f_{m,n}(x,y) = \sin(m\pi x)\sin(n\pi y), \text{ where } m,n \geq 1$$

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So the smallest eigenvalue $\lambda_{\min}$ is $2\pi^2$.

First four Dirichlet eigenvalues for $-\Delta$ on the square in $\mathbb{R}^2$

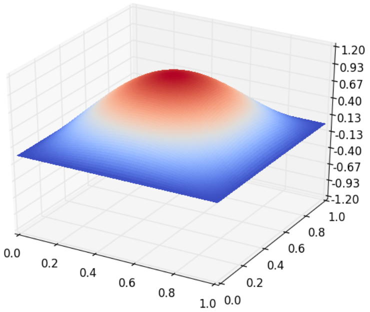

Figure: An eigenfunction $f(x, y) = \sin(\pi x)\sin(\pi y)$ of $-\Delta$ for the smallest Dirichlet eigenvalue $2\pi^2$.

These represent vibrations of a square drum held taut at its boundary.

First four Dirichlet eigenvalues for $-\Delta$ on the square in $\mathbb{R}^2$

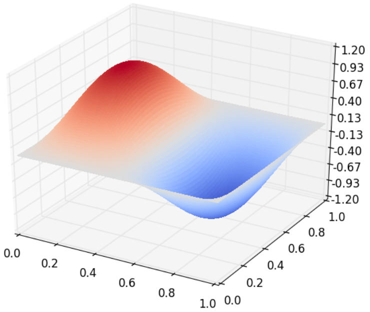

Figure: An eigenfunction $f(x, y) = \sin(2\pi x)\sin(\pi y)$ of $-\Delta$ for the second smallest Dirichlet eigenvalue $5\pi^2$.

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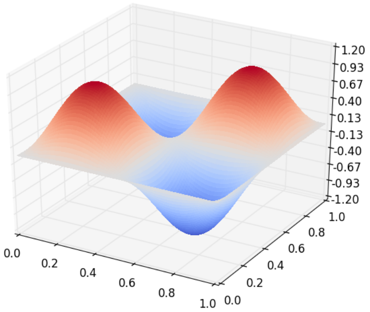

Figure: An eigenfunction $f(x, y) = \sin(2\pi x)\sin(2\pi y)$ of $-\Delta$ for the third smallest Dirichlet eigenvalue $8\pi^2$.

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First four Dirichlet eigenvalues for $-\Delta$ on the square in $\mathbb{R}^2$

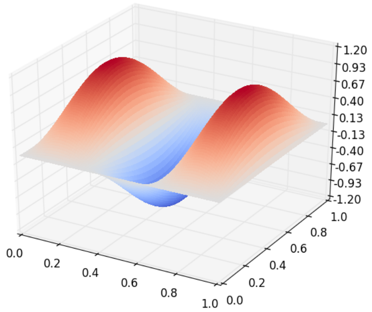

Figure: An eigenfunction $f(x, y) = \sin(3\pi x)\sin(\pi y)$ of $-\Delta$ for the fourth smallest Dirichlet eigenvalue $10\pi^2$.

These represent vibrations of a square drum held taut at its boundary.

Minimax principle

These graphs lead us to an interesting observation.

Minimax principle

Any eigenfunction of $-\Delta$ on Ω for the smallest Dirichlet eigenvalue $\lambda_{\min}$ does not change sign.

An application of Rayleigh's principle

As a separate note about these eigenfunctions, recall Rayleigh's principle for a symmetric matrix

$$\lambda_{\min} = \min_{v \neq 0, v \in V} \frac{\langle Av, v \rangle}{\langle v, v \rangle}$$

Rayleigh's principle applied to the Laplacian (which is symmetric as an operator) leads to the following **comparison theorem**.

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Comparison theorem

If $\Omega \subset \tilde{\Omega}$ are sets in $\mathbb{R}^n$, then if $\lambda_{\min,\Omega}$ and $\lambda_{\min,\tilde{\Omega}}$ represent the minimum Dirichlet eigenvalues for $-\Delta$ on Ω and $\tilde{\Omega}$ respectively, then

$$\lambda_{\min,\tilde{\Omega}} \leq \lambda_{\min,\Omega}$$

In other words, bigger drums vibrate at lower pitches.

Table of Contents

Definitions

The heat equation is what is known as an **evolution equation**. This means we consider solutions to be functions $u(t, x)$, where $t, x \in \mathbb{R}$. In other words, the heat equation describes a time process.

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So we can view $u(x, t)$ in two ways.

- u can be seen as a function on $\mathbb{R}^2$, (t, x).
- Or u can be seen as a function that for each value of $t \in \mathbb{R}$, gives a heat distribution in x that sends $\mathbb{R}$ to $\mathbb{R}$.

The heat equation

$$\partial_t u(t,x) - \Delta u(t,x) = 0$$

We consider a solution $u(t,x)$ that satisfies an initial condition $u(0,x) = f(x)$, which describes a starting distribution of heat.

The heat equation

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We consider a solution $u(t,x)$ that satisfies an initial condition $u(0,x) = f(x)$, which describes a starting distribution of heat.

Remark

The steady state to the heat equation is an equilibrium state of Laplace's equation. If u is a steady state, then the change in time is zero, so the heat equation reduces to $-\Delta u = 0$, which is Laplace's equation.

A brief remark on how to solve it

Although the heat equation can be solved using Fourier series, it is best solved by a tool called the Fourier transform.

The Fourier transform

The Fourier transform $\mathcal{F}$ is defined by

$$\mathcal{F}(f) = \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$$

In some special cases (for Schwartz functions), the Fourier transform has an inverse Fourier transform, given by

$$\mathcal{F}^{-1}(g) = g^{\vee}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)e^{ix\xi}d\xi$$

A brief remark on how to solve it

Taking the spatial Fourier transform of the heat equation gives

$$\partial_t u(t,x) - \Delta u(t,x) = 0 \Longrightarrow \partial_t \widehat{u}(t,\xi) + \xi^2 \widehat{u}(t,\xi) = 0$$

$$u(0,x) = f(x) \Longrightarrow \widehat{u}(0,\xi) = \widehat{f}(\xi)$$

This is an ordinary differential equation that can be easily solved as

$$\widehat{u}(t,\xi) = \widehat{f}(\xi)e^{-\xi^2 t}$$

Taking the inverse Fourier transform, we get the following formula for the solution the heat equation.

Solution to the heat equation

Solution to the heat equation

The solution to $\partial_t u - \Delta u = 0$ with $u(0,x) = f(x)$ as an initial condition is

$$u(t,x) = f(x) * K(x,t) = \int_{-\infty}^{\infty} f(x-y)K(y,t)dy$$

where $K(x,t)$ is the heat kernel

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

The $*$ operation defined above is called convolution.

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If you have taken statistics, you might notice that the heat kernel is a **Gaussian**. The Gaussian is special since it is its own Fourier transform.

The heat kernel

The **heat kernel** gives the heat distribution where the initial condition is that $u(0,x)=\delta(x)$, where $\delta(x)$ is what is called a **Dirac delta function**. This is a "function" that is zero everywhere, except for an infinite peak at $x=0$.

The heat kernel

The **heat kernel** gives the heat distribution where the initial condition is that $u(0, x) = \delta(x)$, where $\delta(x)$ is what is called a **Dirac delta function**. This is a "function" that is zero everywhere, except for an infinite peak at $x = 0$.

So you can think of the heat kernel as the distribution of heat over time if you light a match at a single point in space, and there is no other heat source.

The heat kernel

The **convolution operation** that gives the solution to the heat equation generally tends to “smooth out” functions over time, which can be seen from the ModHeatEq.gif below.

You can see that the heat equation solutions have infinite speed of propagation. Changes in heat are instantly registered everywhere.

HeatKernel.gif

ModHeatEq.gif (Solution to the heat equation for initial function $f(x) = \frac{sin(3x)}{1+x^2}$)

Table of Contents

Definitions

The wave equation looks like the heat equation, except it now has two time derivatives. This subtle difference makes a huge difference, as we will see.

Definition

The wave equation is given by

$$\partial_t^2 u - \Delta u = 0$$

with initial conditions $u(0, x) = f(x)$ and $\partial_t u(0, x) = g(x)$. So we are given an initial configuration of the wave and an initial speed at every point. We are solving for $u(t, x)$.

The partial differential operator $\partial_t^2 - \Delta$ is called the D'Alembertian, denoted by $\Box$. So the wave equation can be written as $\Box u = 0$, $u(0, x) = f(x)$, $\partial_t u(0, x) = g(x)$.

Solution to the wave equation

We will consider $u(x, t)$ where t and x are real numbers. So the wave equation is just

$$\partial_t^2 u(x, t) - \partial_x^2 u(x, t) = 0$$

In this case, there is a closed form expression for the solution.

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In this case, there is a closed form expression for the solution.

Solution to the wave equation in one spatial dimension

$$u(x, t) = \frac{f(x - t) + f(x + t)}{2} + \frac{1}{2} \int_{-t}^{t} g(x + s) ds$$

A brief remark on how to solve it

We quickly outline how to solve it.

- Note that the equation is the same as $(\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$.
- Note that $\partial_t - \partial_x$ is a directional derivative in the $(1, -1)$ direction. So $(\partial_t + \partial_x)u$ is constant along lines of slope $(1, -1)$.
- Use the fundamental theorem of calculus and the fact that $\partial_t + \partial_x$ is a directional derivative in the $(1, 1)$ direction to solve for u.

Huygen's principle

Recall the solution for the wave equation

$$u(x,t) = \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{-t}^{t} g(x+s)ds$$

where $f(x)$ and $g(x)$ represent the initial configuration and velocity of the wave at $t = 0$. Note that u only depends on the value at f at $x - t$ and $x + t$ and the values of g between $x - t$ and $x + t$.

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Recall the solution for the wave equation

$$u(x,t) = \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{-t}^{t} g(x+s)ds$$

where $f(x)$ and $g(x)$ represent the initial configuration and velocity of the wave at $t = 0$. Note that u only depends on the value at f at $x - t$ and $x + t$ and the values of g between $x - t$ and $x + t$.

Huygen's principle

The value $u(x,t)$ of a wave only depends on the values of f and g between $x - t$ and $x + t$.

In particular, given initial data at some point x_0, this initial data at x_0 cannot influence the behavior of the wave at $u(x,t)$ until times past $t = |x - x_0|$. This is called finite speed of propagation.

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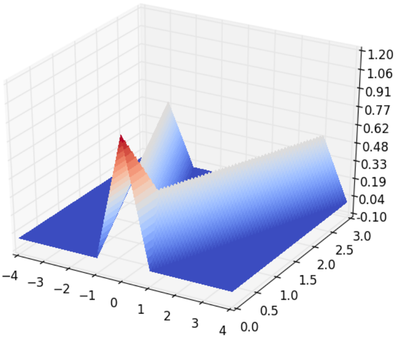
In particular, it takes time for a disturbance to travel to another point on a wave.

A visual illustration of Huygen's principle

WaveEq.gif

Let us look at the solution to the wave equation for

$$f(x) = 1 - x, 0 \le x \le 1 \qquad f(x) = 1 + x, -1 \le x \le 0, \qquad g(x) = 0$$



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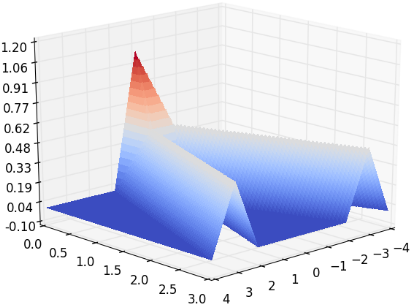
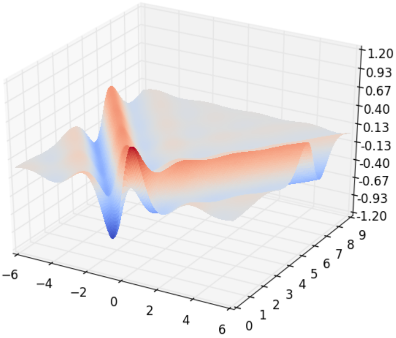


A visual illustration of Huygen's principle

WaveSine.gif

Let us look at the solution to the wave equation for

$$f(x) = \frac{\sin(2x)}{1 + x^2} \qquad g(x) = 0$$



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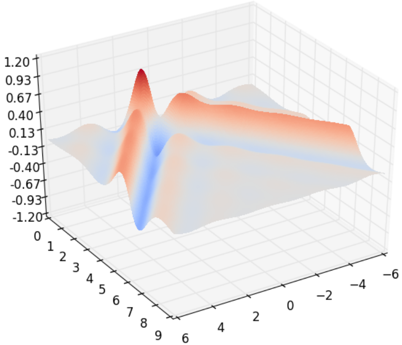

Traveling wave solutions

Finally, we note that the wave equation admits traveling wave solutions, which are solutions that travel without changing shape.

Traveling waves

For any smooth functions h_1 and h_2, the wave equation admits traveling wave solutions of the form

$$u(x,t) = h_1(x-t) + h_2(x+t)$$

$h_1(x-t)$ is a wave moving to the right with velocity 1, and $h_2(x+t)$ is a wave moving to the left with velocity 1.

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Traveling wave solutions to wave equations are of special interest, and give rise to stationary wave phenomena, such as that of solitons in water waves, which are water waves that retain their shape while traveling.

Traveling waves in opposite directions pass each other without interacting

Travel.gif

Acknowledgements

All gif animations were made using `imgflip.com`. All images were made using Matplotlib.

References and Additional Reading

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