Math 1A: Midterm Review Problems Solutions

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Question 1

We find the domain of

$$f(x) = \frac{\ln(x^2 - 4)}{x^2}$$

We must have that $x^2 - 4 > 0$. Factoring, we have that $(x-2)(x+2) > 0$. So then that means $x > 2$ or $x < -2$. We also have $x^2 \neq 0$, so that $x \neq 0$. However, this is already excluded by the condition $x^2 - 4 > 0$.

So the domain is $x > 2$ or $x < -2$, also written as $(-\infty, -2) \cup (2, \infty)$.

Question 2

First, find the domain of

$$f(x) = \frac{x}{1 - \tan(x)}$$

We must have that $1 - \tan(x) \neq 0$, so that $\tan(x) \neq 1$. Then, that means that

$$x \neq \frac{\pi}{4} + \pi k$$

for integers k. So the domain of f is

$$x \neq \frac{\pi}{4} + \pi k$$

for integers k.

The domain of

$$g(x) = \frac{\sqrt{x}}{1 - \tan(x)}$$

is indeed different, because the square root introduces the restriction that $x \geq 0$. So the domain is now

$$x \neq \frac{\pi}{4} + \pi k$$

for nonnegative integers k. (So $x \neq \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, ...$).

Question 3

We compute

$$\lim_{x\to 2}\frac{\sqrt{x^2+5}}{\cos(\pi x)} = \frac{\lim_{x\to 2}\sqrt{x^2+5}}{\lim_{x\to 2}\cos(\pi x)} = \frac{\sqrt{\lim_{x\to 2}(x^2+5)}}{\lim_{x\to 2}\cos(\pi x)}$$

$$= \frac{\sqrt{\lim_{x\to 2}x^2+\lim_{x\to 2}5}}{\lim_{x\to 2}\cos(\pi x)} = \frac{\sqrt{4+5}}{1} = 3$$

Question 4

We compute

$$\lim_{h\to 1}\frac{\sqrt{h^2+h+2}-2}{h-1}$$

$$= \lim_{h\to 1}\frac{\sqrt{h^2+h+2}-2}{h-1}\left(\frac{\sqrt{h^2+h+2}+2}{\sqrt{h^2+h+2}+2}\right) = \lim_{h\to 1}\frac{(h^2+h+2)-4}{(h-1)(\sqrt{h^2+h+2}+2}$$

$$= \lim_{x\to 1}\frac{h^2+h-2}{(h-1)(\sqrt{h^2+h+2}+2} = \lim_{h\to 1}\frac{(h-1)(h+2)}{(h-1)(\sqrt{h^2+h+2}+2)}$$

$$= \lim_{h\to 1}\frac{h+2}{\sqrt{h^2+h+2}+2} = \frac{3}{\sqrt{4}+2} = \frac{3}{4}$$

Question 5

We have that

$$(f\circ f)(x) = \frac{1}{\frac{1}{x(x-4)}\left(\frac{1}{x(x-4)}-4\right)}$$

We have that the domain of f is $x\neq 0$, $x\neq 4$. So to find the domain of $f\circ f$, we need to find when

$$\frac{1}{x(x-4)} = 0$$

or

$$\frac{1}{x(x-4)} = 4$$

The first equation has no solutions, as one can check. The second equation has solutions

$$x(x-4) = \frac{1}{4}$$

$$4x^2-16x-1 = 0$$

$$x = \frac{16\pm\sqrt{256+16}}{8} = 2\pm\frac{\sqrt{272}}{8} = 2\pm\frac{\sqrt{17}}{2}$$

So the domain of $f \circ f$ is

$$x \neq 0, 4, 2 + \frac{\sqrt{17}}{2}, 2 - \frac{\sqrt{17}}{2}$$

We have that

$$(g \circ g)(x) = 2\cos(2\cos(x))$$

The domain of $g \circ g$ is all real numbers.

We have that

$$(f \circ g)(x) = \frac{1}{2\cos(x)(2\cos(x) - 4)}$$

To find the domain, we want to find when

$$2\cos(x) = 0$$

and

$$2\cos(x) = 4$$

and exclude these values. The first equation has solutions $x = \frac{\pi}{2} + k\pi$ for integers k, and the second equation (which is equivalent to $\cos(x) = 2$) has no solutions (since the values of cosine are between -1 and 1 inclusive). So the domain of $f \circ g$ is

$$x \neq \frac{\pi}{2} + k\pi$$

for integers k.

Finally, we consider

$$(g \circ f)(x) = 2\cos\left(\frac{1}{x(x-4)}\right)$$

The domain of $g \circ f$ is $x \neq 0, 4$ (from the restriction for the domain of f).

Question 6

We note that $y = |x|$, $y = 3x$, $y = \sqrt{x}$, and $y = 2$ are all continuous functions, and since the sum of continuous functions is continuous, we have that $y = |x| + 3x + \sqrt{x} + 2$ is a continuous function. Therefore,

$$\lim_{x \to 0} \left(|x| + 3x + \sqrt{x} + 2\right) = |0| + 3(0) + \sqrt{0} + 2 = 2$$

Question 7

Remember that arccos takes values in $[0, \pi]$. We have that

$$\arccos\left(\cos\left(\frac{10\pi}{3}\right)\right) = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Next, we have that

$$\log_3\left(\tan\left(\frac{7\pi}{6}\right)\right) = \log_3\left(\frac{1}{\sqrt{3}}\right) = -\frac{1}{2}$$

Finally, we have that

$$\sin(e^{\ln(\pi)-\ln(3)}) = \sin\left(e^{\ln(\pi/3)}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Question 8

We use the Squeeze Theorem and the limit laws. We note that

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

so then

$$-\sqrt{x} \leq \sqrt{x}\,\cos\left(\frac{1}{x}\right) \leq \sqrt{x}$$

and since $\lim_{x\to 0} -\sqrt{x} = \lim_{x\to 0}\sqrt{x} = 0$, by the Squeeze Theorem, we have that

$$\lim_{x\to 0}\ \sqrt{x}\,\cos\left(\frac{1}{x}\right) = 0$$

In addition,

$$-1 \leq \sin\left(\frac{1}{\sqrt{x}}\right) \leq 1$$

and we can check that

$$-|x| \leq x\,\sin\left(\frac{1}{\sqrt{x}}\right) \leq |x|$$

In addition, $\lim_{x\to 0} -|x| = \lim_{x\to 0}|x| = 0$, and so by the Squeeze Theorem,

$$\lim_{x\to 0}\ x\,\sin\left(\frac{1}{\sqrt{x}}\right) = 0$$

Then using the limit laws,

$$\lim_{x\to 0}\left[\sqrt{x}\,\cos\left(\frac{1}{x}\right) + x\,\sin\left(\frac{1}{\sqrt{x}}\right)\right] = \lim_{x\to 0}\sqrt{x}\cos\left(\frac{1}{x}\right) + \lim_{x\to 0} x\sin\left(\frac{1}{\sqrt{x}}\right) = 0+0 = 0$$

Question 9

We have that $f(0) = 1$. To make f continuous at $x = 0$, we want $\lim_{x\to 0} f(x) = f(0) = 1$. Therefore, we need

$$\lim_{x\to 0^-} f(x) = 1$$

So we need

$$\lim_{x\to 0^-} \frac{ae^x}{1+e^x} = \frac{a}{2} = 1$$

So to have f be continuous, we need $a = 2$.

Question 10

We first have that

$$\lim_{x\to\infty} \left[\frac{2x^2+1}{3x^2+2x+\sqrt{x}+1}\right] = \lim_{x\to\infty} \left[\frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+2x+\sqrt{x}+1}{x^2}}\right] = \lim_{x\to\infty} \left[\frac{2+\frac{1}{x^2}}{3+\frac{2}{x}+\frac{1}{x^{3/2}}+\frac{1}{x^2}}\right] = \frac{2}{3}$$

Next, we have that as $x \to -\infty$, $\arctan(x)$ gets closer and closer to $-\pi/2$ but always still greater than $-\pi/2$. So as $x \to -\infty$, $\pi/2 + \arctan(x)$ is positive and getting smaller and smaller. Therefore, as $x \to -\infty$,

$$\frac{1}{\frac{\pi}{2} + \arctan(x)}$$

becomes 1 divided by something very small and positive. So

$$\lim_{x\to-\infty} \left[\frac{1}{\frac{\pi}{2} + \arctan(x)}\right] = \infty$$

Finally, for the last limit, we plug in $x = 2$, and get $1/0$, which implies that we are looking at an infinite limit. We have that as $x \to 2^+$, the numerator is around 1 and $(x-2)^5$ becomes small and positive. When $x \to 2^-$, the numerator is around 1 and $(x-2)^5$ becomes small and negative. So we have

$$\lim_{x\to 2^+} \frac{\cos(\pi x)}{(x-2)^5} = \infty$$

$$\lim_{x\to 2^-} \frac{\cos(\pi x)}{(x-2)^5} = -\infty$$

so since the limits are not both the same sign of infinity, the only thing we can say is that

$$\lim_{x\to 2} \left[\frac{\cos(\pi x)}{(x-2)^5}\right]$$

does not exist.

Question 11

We want to use the fact that

$$|x| = x \text{ if } x \geq 0$$

$$|x| = -x \text{ if } x < 0$$

To consider whether the limit exists, we consider the left and right hand limits and see if they agree.

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

Since the left and right hand limits do not agree, we have that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

The right and left hand limits of the second limit actually do exist and agree with each other.

$$\lim_{x\to 0^+} \frac{|x|+1}{x+1} = \lim_{x\to 0^+} \frac{x+1}{x+1} = 1$$

$$\lim_{x\to 0^-} \frac{|x|+1}{x+1} = \lim_{x\to 0^-} \frac{-x+1}{x+1} = 1$$

So therefore,

$$\lim_{x\to 0} \frac{|x|+1}{x+1} = 1$$

Question 12

Continuity at $x = 0$ means that $\lim_{x\to 0} f(x) = f(0)$. Since $f(0) = 1$, it suffices to show that

$$\lim_{x\to 0} (x^2+1) = 1$$

Consider any $\epsilon > 0$. We want to find a $\delta > 0$ so that if $0 < |x - 0| < \delta$, then $|(x^2+1) - 1| < \epsilon$. Equivalently, we want to find a $\delta > 0$ so that if $0 < |x| < \delta$, then $|x^2| < \epsilon$.

We see that $\delta = \sqrt{\epsilon}$ works. We check this. If $0 < |x| < \delta$, then $|x| < \sqrt{\epsilon}$, and then $|x^2| < \epsilon$ (since $|x^2| = |x|^2$).

Question 13

Note that $\arctan(x)$, e^{3x}, and -1 are all continuous functions, so $\arctan(x)+(e^{3x})-1$ is a continuous function. Since e^x is also continuous, we have that

$$e^{\arctan(x)+(e^{3x})-1}$$

is continuous since it is the composition of continuous functions. So then using continuity, we can evaluate the limit by just plugging in $x = 0$. So

$$\lim_{x\to 0} e^{\arctan(x)+(e^{3x})-1} = e^{\arctan(0)+e^0-1} = e^0 = 1$$

Question 14

As $x \to 0^+$, we have that $\cos(x^2)$ is basically around $1 = \cos(0)$, and $\frac{1}{\sqrt{x}}$ is going to positive infinity. Therefore,

$$\lim_{x\to 0^+} f(x) = \infty$$

So the limit does not exist, and is positive infinity.

Question 15

We start with $f(x) = \sqrt{x-2} + 3$.

- We first shift down by 3. Then, we get the graph of $y = \sqrt{x-2}$.
- We then shift to the left by 4. Then, we get the graph of $y = \sqrt{(x+4)-2} = \sqrt{x+2}$.
- We then reflect across the x axis. We then get the graph of $y = -\sqrt{x=2}$.
- We then use a vertical stretch by 2. We finally end up with the graph of $y = -2\sqrt{x+2}$.