Math 54 Midterm 3 (Practice 4)

Jeffrey Kuan

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Name: ____

SSID: ____

Instructions:

- This exam is **120 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Part (a)

Show that $T(x_1, x_2, x_3) = (-x_3, x_1, -x_2)$ is an isometry on $\mathbb{R}^3$. [15 points]

Part (b)

Show that there is no value of a such that

$$T(x_1, x_2) = (ax_1 + 3x_2, ax_1 + x_2)$$

is an isometry on $\mathbb{R}^2$.

Problem 2 (30 points)

Consider the linear operator $T : M_{2\times 2} \to M_{2\times 2}$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Calculate the determinant and trace of T. Calculate $\mathrm{char}_T(x)$.

Problem 3 (30 points)

Short proofs. [15 points each]

Part (a)

Define a **negative definite matrix** to be a symmetric n by n matrix A such that $\langle Av, v \rangle < 0$ for all nonzero $v \in \mathbb{R}^n$. Prove that every diagonal entry of a negative definite matrix is negative.

Part (b)

Find, with proof, the distinct eigenvalues of a general lower triangular n by n matrix and their algebraic multiplicities.

Problem 4 (30 points)

Show that $\mathcal{C} = \{(1,1,1,0), (-2,1,1,0), (0,-1,1,0), (0,0,0,1)\}$ is an orthogonal basis for $\mathbb{R}^4$. Consider the linear transformation $T : P_3 \to \mathbb{R}^4$ given by

$$T(p(x)) = \begin{bmatrix} p(-10) \\ p(-1) \\ p(1) \\ p(10) \end{bmatrix}$$

Find $[T]_{\mathcal{B}\to\mathcal{C}}$ where $\mathcal{B} = \{1, x, x^2, x^3\}$ is the standard basis for P_3.

Problem 5 (20 points)

Find all least squares solutions to the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 2 & -3 & 4 & 0 \\ -2 & -2 & -4 & 6 & -8 & 0 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. What is the least squares error?

Problem 6 (30 points)

Let $\{v_1, v_2\}$ be an orthonormal basis for $\mathbb{R}^2$. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator and that $\{Tv_1, Tv_2\}$ is also an orthonormal basis for $\mathbb{R}^2$. Prove that T is an isometry on $\mathbb{R}^2$.

Problem 7 (30 points)

Suppose that A is an arbitrary 2 by 2 matrix that has trace 1 and determinant -6.

Part (a)

Find the eigenvalues of A. [20 points]

Part (b)

Is A diagonalizable? Justify your answer. [10 points]

END OF EXAM