Math 54 Midterm 3 (Practice 3)

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Name: ____

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Instructions:

- This exam is **120 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Consider the Bilaplacian $\Delta\Delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, given by $(\Delta\Delta)(f(x)) = f''''(x)$.

Part (a)

Show that every positive number is an eigenvalue. [15 points]

Part (b)

Is $\lambda = 0$ an eigenvalue of $\Delta\Delta$? If yes, find a basis for the eigenspace corresponding to $\lambda = 0$. [15 points]

Problem 2 (30 points)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & -2 \\ -2 & -2 & -1 \end{bmatrix}$$

Find the characteristic polynomial of A, all eigenvalues (and their algebraic and geometric multiplies), and all eigenvectors. If A is diagonalizable, diagonalize A. [20 points]

Part (b)

Calculate $(A^{-1})^{100}v$, where $v = (1, 1, 1)$. [10 points]

Problem 3 (30 points)

Short proofs. [15 points each]

Part (a)

Let V be an inner product space (not necessarily $\mathbb{R}^n$). Show that if v_1, v_2, and v_3 comprise an orthogonal set in V, then

$$||v_1 + v_2 + v_3||^2 = ||v_1||^2 + ||v_2||^2 + ||v_3||^2$$

Part (b)

Prove that for every positive definite matrix A, there exists a positive definite matrix B such that $B^4 = A$.

Problem 4 (30 points)

Part (a)

Find the distance from the point $(2, 1, 1)$ to the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 0 & -1 \\ 2 & -2 & 2 \\ -1 & 3 & -1 \end{bmatrix}$$

[20 points]

Part (b)

Find the point in the nullspace of the matrix A (where A is the matrix in part (a)) that is closest to $(2, 1, 1)$.

Problem 5 (30 points)

Part (a)

Let A be a matrix with a nontrivial nullspace. Find an eigenvalue of A. [15 points]

Part (b)

Let A be a matrix such that the entries in every row sum to a fixed number k. Find an eigenvalue of A. [15 points]

Problem 6 (30 points)

Perform the Gram-Schmidt process on the vectors $\mathcal{B} = \{e^x, e^{2x}\}$ in $L^2([0,1])$. [30 points]

Problem 7 (20 points)

Consider the matrices

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

First, determine if A is positive definite. Then, determine if B is positive semidefinite. [20 points]

END OF EXAM