Math 54 Midterm 3 (Practice 2)

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Name: ____

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Instructions:

- This exam is **120 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Is A positive definite? Is A positive semidefinite? What is the maximum value of $\frac{\langle Av, v \rangle}{\langle v, v \rangle}$ for a nonzero vector v?

Problem 2 (30 points)

Consider the following linear transformation $T : P_2 \to P_2$ given by

$$T(p(x)) = \frac{d}{dx}((x+1)p(x))$$

Part (a)

Find the determinant of T, the trace of T, all eigenvalues of T, and all eigenvectors of T. [15 points]

Part (b)

Find $T^{100}(x^2)$. [15 points]

Problem 3 (30 points)

Short proofs. [15 points each]

Part (a)

Let V be an inner product space. Show that if $\langle v, u_1 \rangle = \langle v, u_2 \rangle$ for all $v \in V$, then $u_1 = u_2$.

Part (b)

Let V be an inner product space, and let W be a subspace of V. Show that $W^\perp$ is a subspace of V, and that the only vector W in both W and $W^\perp$ is the zero vector.

Problem 4 (30 points)

Show that if $T_1 : V \to V$ and $T_2 : V \to V$ are both isometries, then $T_2 \circ T_1 : V \to V$, defined by $(T_2 \circ T_1)(v) = T_2(T_1(v))$ is an isometry too.
[10 points]

Part (b)

Find conditions on a, b, and c (if any exist) such that

$$T(x_1, x_2, x_3) = (ax_1 + x_2 + x_3, -x_1 + bx_2 + 2x_3, -2x_1 + cx_3)$$

is an isometry. [20 points]

Problem 5 (30 points)

Consider the matrix
$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Part (a)

Find an orthonormal basis for $\mathrm{Col}(A)$. [25 points]

Part (b)

Find the dimension of $(\mathrm{Col}(A))^{\perp}$. [5 points]

Problem 6 (30 points)

Find all a and b such that

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ a & b & 0 \end{bmatrix}$$

is diagonalizable.

Problem 7 (20 points)

Part (a)

Find an example of a matrix that has no real eigenvalues, or show that it does not exist. [6 points]

Part (b)

Find a matrix with all entries positive that is not positive definite, or show that it does not exist. [7 points]

Part (c)

Find a matrix that has only one distinct eigenvalue but is not diagonalizable, or show that it does not exist. [7 points]

END OF EXAM