Math 54 Midterm 3 (Practice 1)

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Name: ____

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Instructions:

- This exam is **120 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Consider the following matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Find the geometric and algebraic multiplicity of every eigenvalue and a basis for each eigenspace. What is the characteristic polynomial of A? Is A diagonalizable? If so, diagonalize it.

Problem 2 (20 points)

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix}$$

Find all least squares solutions to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$. In addition, find the least squares error.

Problem 3 (30 points)

Short proofs. [15 points each]

Part (a)

Prove that if u and v are orthonormal vectors in an inner product space V (not necessarily $\mathbb{R}^n$) and a and b are real constants, then

$$||au + bv|| = \sqrt{a^2 + b^2}$$

Part (b)

Show that if $v_1, v_2, ..., v_n$ form an orthonormal set in an inner product space V (not necessarily $\mathbb{R}^n$), then $v_1, v_2, ..., v_n$ are linearly independent.

Problem 4 (30 points)

Define a **negative definite matrix** to be a symmetric n by n matrix A such that $\langle Av, v\rangle < 0$ for all nonzero vectors $v \in \mathbb{R}^n$.

Part (a)

Show that if A is a negative definite n by n matrix, then A has n negative eigenvalues. [15 points]

Part (b)

Determine whether the matrix

$$A = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

is negative definite. (Hint: If A is negative definite, what is true about $-A$?) [15 points]

Problem 5 (30 points)

Part (a)

Find a unit vector for the vector $f(x) = x$ in $L^2([0,1])$. [5 points]

Part (b)

Let W be the subspace of polynomials in P_2 that are orthogonal to $f(x) = x$ with respect to the $L^2([0,1])$ inner product. Find $\dim(W)$ and find a basis for W. [25 points]

Problem 6 (30 points)

Let W consist of the set of points that lie on the intersection of the planes $x_1+x_2-x_3-2x_4=0$ and $x_1-2x_3+x_4=0$ in $\mathbb{R}^4$.

Part (a)

Find an orthonormal basis for W. [20 points]

Part (b)

Write $v = (20, 10, -10, -20)$ as a sum of vector in W and a vector in $W^{\perp}$. Is such a decomposition of v as a sum of vector in W and a vector in $W^{\perp}$ unique? [10 points]

Problem 7 (30 points)

Suppose that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ has $T(1,1,1) = 2$, $T(1,-1,0) = 1$, and $T(0,1,-1) = -1$. You may use the fact that $\{(1,1,1),(1,-1,0),(0,1,-1)\}$ is a basis for $\mathbb{R}^3$.

Part (a)

Calculate $T(4,26,29)$, $T(7,31,19)$, and T(-100, -20, 40). [20 points]

Part (b)

Find a basis for $(\ker(T))^{\perp}$. [10 points] (Hint: This should be quick if you did part (a) in an efficient way.)

END OF EXAM