Math 54 Midterm 3

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August 5, 2019

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SSID: Answer Key

Instructions:

- This exam is **120 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (25 points)

Consider the linear transformation $T : P_3 \to P_3$ given by

$$T(p(x)) = (x^2 + 1)p''(x)$$

Find the characteristic polynomial of T, the determinant of T, the trace of T, and all eigenvalues and eigenspaces of T (if any exist). Is T diagonalizable?

$T(1) = 0 \quad T(x) = 0 \quad T(x^2) = (x^2+1)2 = 2x^2+2$

$T(x^3) = (x^2+1)(6x) = 6x^3 + 6x$

$\mathcal{B} = \{1, x, x^2, x^3\}$

$$[T]_{\mathcal{B}\to\mathcal{B}} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$\text{char}_T(x) = x^2(x-2)(x-6)$

$\det_T(x) = 0 \cdot 0 \cdot 2 \cdot 6 = 0$

$\text{tr}_T(x) = 0+0+2+6 = 8$

$\lambda = 0, 0, 2, 6$

$\lambda = 0 \quad \text{nullspace}\left(\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\right) \quad s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\Downarrow$

$\boxed{\lambda = 0 \quad s + tx}$

$\lambda = 2 \quad \text{nullspace}\left(\begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\right) \quad s\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \boxed{\lambda = 2 \quad s(1+x^2)}$

$\lambda = 6 \quad \text{nullspace}\left(\begin{bmatrix} -6 & 0 & 2 & 0 \\ 0 & -6 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \quad s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \boxed{\lambda = 6 \quad s(x + x^3)}$

Yes, T is diagonalizable.

Problem 2 (30 points)

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 0 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 3 & 0 & -2 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Part (a)

Find an orthonormal basis for Col(A). What is the dimension of $(\text{Col}(A))^{\perp}$? [15 points]

$$A \to \begin{bmatrix} 1 & 1 & -1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col(A): $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$ $\quad \boxed{w_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}$

$$v_2 - \langle v_2, w_1 \rangle w_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} - \frac{1}{3}(3) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\boxed{w_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}}$$

Part (b)

Find all least squares solutions to $A\mathbf{x} = \mathbf{b}$ where

$$\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

and A is the matrix above. [15 points]

$$\text{proj}_{\text{Col}(A)} b = c_1 w_1 + c_2 w_2 \qquad c_1 = \langle b, w_1 \rangle = \frac{1}{\sqrt{3}} \cdot 6$$

$$c_2 = \langle b, w_2 \rangle = \frac{1}{\sqrt{3}} (-3)$$

$$= \frac{6}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{-3}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -1 & 2 & 0 & -1 & 2 \\ 1 & 0 & -1 & 1 & 0 & 0 & 3 \\ 1 & 2 & -1 & 3 & 0 & -2 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & 0 & -1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underset{3}{0} & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\hat{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

Problem 3 (30 points)

Short proofs. [15 points each]

Part (a)

Let V be any inner product space (not necessarily $\mathbb{R}^n$). Prove the following alternate form of the **polarization identity**

$$\langle u, v\rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2)$$

$$\frac{1}{4}(\|u+v\|^2 - \|u-v\|^2) = \frac{1}{4}(\langle u+v, u+v\rangle - \langle u-v, u-v\rangle)$$

$$= \frac{1}{4}(\langle u, u+v\rangle + \langle v, u+v\rangle - \langle u, u-v\rangle + \langle v, u-v\rangle)$$

$$= \frac{1}{4}(\cancel{\langle u,u\rangle} + \langle u,v\rangle + \langle v,u\rangle + \cancel{\langle v,v\rangle} - \cancel{\langle u,u\rangle} + \langle u,v\rangle$$
$$+ \langle v,u\rangle - \cancel{\langle v,v\rangle})$$

$$= \frac{1}{4}(\langle u,v\rangle + \langle u,v\rangle + \langle u,v\rangle + \langle u,v\rangle)$$

$$= \langle u,v\rangle \checkmark$$

Part (b)

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. For any vector $w \in \mathbb{R}^n$, show that the function $\ell(v) = \langle Tv, w\rangle$ is a linear functional on $\mathbb{R}^m$. Use this fact to show that for every $w \in \mathbb{R}^n$, there exists a unique vector $u \in \mathbb{R}^m$ such that

$$\langle Tv, w\rangle = \langle v, u\rangle$$

holds for every $v \in \mathbb{R}^m$. You may use any theorem we have proved in class.

Need to check that $\ell(v) = \langle Tv, w\rangle$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}$.

Check linearity: $\ell(c_1v_1 + c_2v_2) = \langle T(c_1v_1 + c_2v_2), w\rangle$

$$= \langle c_1Tv_1 + c_2Tv_2, w\rangle$$

$$= c_1\langle Tv_1, w\rangle + c_2\langle Tv_2, w\rangle$$

$$= c_1\ell(v_1) + c_2\ell(v_2) \checkmark$$

So ℓ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}$ so ℓ is a linear functional on $\mathbb{R}^m$. So by the Riesz Representation Theorem, there exists a unique vector $u \in \mathbb{R}^m$ such that

$$\langle Tv, w\rangle = \ell(v) = \langle v, u\rangle$$

for all $v \in \mathbb{R}^m$.

Problem 4 (35 points)

Consider the symmetric matrix

$$A = \begin{bmatrix} 2x & 1 & x \\ 1 & 1 & 0 \\ x & 0 & 1 \end{bmatrix}$$

Part (a)

Find all real numbers x (if any exist) such that A is positive definite. [20 points]

Use Sylvester's Criterion.

$$|2x| = 2x \qquad \begin{vmatrix} 2x & 1 \\ 1 & 1 \end{vmatrix} = 2x-1 \qquad \begin{vmatrix} 2x & 1 & x \\ 1 & 1 & 0 \\ x & 0 & 1 \end{vmatrix} = x(-x)+1(2x-1)$$

$$= -(x^2-2x+1)$$

$$= -(x-1)^2$$

Since $-(x-1)^2$ is always ≤ 0, for no value of x is A positive definite.

Part (b)

Orthogonally diagonalize this matrix A when $x = 1$. [15 points]

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \begin{vmatrix} 2-x & 1 & 1 \\ 1 & 1-x & 0 \\ 1 & 0 & 1-x \end{vmatrix}$$

$$= 1(x-1) + (1-x)(2-3x+x^2-1)$$

$$= 1(x-1) + (x-1)(-x^2+3x-1)$$

$$= (x-1)(-x^2+3x) = -x(x-3)(x-1)$$

$\lambda = 0$ $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ Basis: $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$

$\lambda = 1$ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ Basis: $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$\lambda = 3$ $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ Basis: $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$

$$\boxed{A = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}}$$

Problem 5 (25 points)

Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 & 6 \end{bmatrix}$$

- Is A positive definite? Justify your answer.
- Prove that A has an eigenvalue that is greater than or equal to 9.

(Hint: Consider the vectors $v = (1, -1, 0, ..., 0)$ and $w = (1, 1, 1, ..., 1)$.)

For $v = (1, -1, 0, \_, 0)$,

$$Av = (0, 0, -2, 0, \_, 0)$$

So $\frac{\langle Av, v \rangle}{\langle v, v \rangle} = \frac{0}{2} = 0$

For $w = (1, 1, \_, 1)$,

$$Aw = (2, 4, 7, 10, 13, 16, 12)$$

So $\frac{\langle Aw, w \rangle}{\langle w, w \rangle} = \frac{64}{7}$

So since $\lambda_{min} = \min_{v \neq 0, v \in \mathbb{R}^n} \frac{\langle Av, v \rangle}{\langle v, v \rangle}$ $\lambda_{max} = \max_{v \neq 0, v \in \mathbb{R}^n} \frac{\langle Av, v \rangle}{\langle v, v \rangle}$

by Rayleigh's principle,

$\lambda_{min} \leq 0$ (so A is not positive definite since not all eigenvalues are positive)

and $\lambda_{max} \geq \frac{64}{7} \geq 9$ so that A has an eigenvalue that is ≥ 9.

Problem 6 (30 points)

Let W be the subspace consisting of points (x, y, z) in $\mathbb{R}^3$ such that $x + 2y - z = 0$.

Part (a)

Find an orthonormal basis for W. [10 points]

$$x + 2y - z = 0$$

$$\left[\begin{array}{ccc|c} 1 & \underset{\text{free}}{\underset{\uparrow}{2}} & \underset{\text{free}}{\underset{\uparrow}{-1}} & 0 \end{array}\right] \quad \text{Basis: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} w_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ w_2 &= \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}} \quad v_2 - \langle v_2, w_1 \rangle w_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}(-2)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Part (b)

Define the projection linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(v) = \mathrm{proj}_W(v)$, where W is defined above. Find the matrix of T with respect to the standard basis $\mathcal{B}$ for $\mathbb{R}^3$, $[T]_{\mathcal{B}\to\mathcal{B}}$. [20 points]

$$\begin{aligned} \mathrm{proj}_W(1,0,0) &= \langle e_1, w_1 \rangle w_1 + \langle e_2, w_2 \rangle w_2 \\ &= \frac{1}{2}(1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3}(-1)\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/6 \\ -1/3 \\ 1/6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathrm{proj}_W(0,1,0) &= \langle e_2, w_1 \rangle w_1 + \langle e_2, w_2 \rangle w_2 \\ &= \frac{1}{2}(0)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \left(\frac{1}{3}\right)(1)\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathrm{proj}_W(0,0,1) &= \langle e_3, w_1 \rangle w_1 + \langle e_3, w_2 \rangle w_2 \\ &= \frac{1}{2}(1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \left(\frac{1}{3}\right)(1)\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 1/3 \\ 5/6 \end{pmatrix} \end{aligned}$$

$$\boxed{[T]_{\mathcal{B}\to\mathcal{B}} = \begin{bmatrix} 5/6 & -1/3 & 1/6 \\ -1/3 & 1/3 & 1/3 \\ 1/6 & 1/3 & 5/6 \end{bmatrix}}$$

Problem 7 (25 points)

Let V be the vector space of infinite sequences of real numbers. Find all eigenvalues and eigenvectors (if any exist) of the linear transformation $T : V \to V$ given by

$$T(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$$

(Hint: Directly apply the definition of eigenvalue and eigenvector. Do not try to use a matrix to solve this.)

Recall λ is an eigenvector if and only if $\ker(T - \lambda I)$ is not just the zero vector (which is the infinite sequence of all 0s).

$$(T - \lambda I)(x_1, x_2, x_3, \_)$$

$$= (0, x_1, x_2, x_3, \_) - \lambda(x_1, x_2, x_3, \_)$$

$$= (-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \_)$$

If $\lambda = 0$, $(T - 0I)(x_1, x_2, x_3, \_)$

$= (0, x_1, x_2, \_)$ and $\ker(T - 0I) = \{0\}$.

$\nearrow$ zero sequence

So $\lambda = 0$ is not an eigenvalue.

If $\lambda \neq 0$, $(T - \lambda I)(x_1, x_2, x_3, \_)$

$= (-\lambda x_1, x_1 - \lambda x_2, x_2 - \lambda x_3, \_)$

To calculate $\ker(T - \lambda I)$,

if $\lambda \neq 0$, $-\lambda x_1 = 0 \Rightarrow x_1 = 0$

$x_1 - \lambda x_2 = 0 \Rightarrow -\lambda x_2 = 0 \Rightarrow x_2 = 0$

$x_2 - \lambda x_3 = 0 \Rightarrow -\lambda x_3 = 0 \Rightarrow x_3 = 0$

$\vdots$

all $x_i = 0$.

END OF EXAM

So $\ker(T - \lambda I) = \{0\}$

$\nearrow$ zero sequence

So any $\lambda \neq 0$ is not an eigenvalue.

So T has no eigenvalues and no eigenvectors.