Math 54 Midterm 3 (Practice 4)

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SSID: Answer Key

Instructions:

- This exam is **120 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Part (a)

Show that $T(x_1, x_2, x_3) = (-x_3, x_1, -x_2)$ is an isometry on $\mathbb{R}^3$. [15 points]

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Check $O = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ is an orthogonal matrix.

$$OO^T = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

(or note cols of O are an ONB)

Part (b)

Show that there is no value of a such that

$$T(x_1, x_2) = (ax_1 + 3x_2, ax_1 + x_2)$$

is an isometry on $\mathbb{R}^2$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} a & 3 \\ a & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$(a,a), (3,1)$ will never be an ONB

since $(3,1)$ does not have length 1.

So for no real a is $\begin{bmatrix} a & 3 \\ a & 1 \end{bmatrix}$ orthogonal matrix.

$\boxed{\text{So no } a \text{ exist.}}$

Problem 2 (30 points)

Consider the linear operator $T : M_{2\times 2} \to M_{2\times 2}$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Calculate the determinant and trace of T. Calculate $\text{char}_T(x)$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$[T]_{\mathcal{B}\to\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\det(T) = (-1)\begin{vmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)(-1)(1) = \boxed{1}$$

$$\text{tr}(T) = 0 + (-1) + 0 + (-1) = \boxed{-2}$$

$$\begin{vmatrix} -x & 1 & 0 & 1 \\ -1 & -1-x & -1 & -1 \\ 0 & 0 & -x & 1 \\ 0 & 0 & -1 & -1-x \end{vmatrix} = (-x)(-1-x)(x+x^2+1) + 1(1)(x+x^2+1)$$

$$= (x+x^2)(x+x^2+1) + (x+x^2+1)$$

$$= \boxed{(x^2+x+1)^2}$$

• Orthogonally diagonalize the 4 by 4 ...

Problem 3 (30 points)

Short proofs. [15 points each]

Part (a)

Define a **negative definite matrix** to be a symmetric n by n matrix A such that $\langle Av, v\rangle < 0$ for all nonzero $v \in \mathbb{R}^n$. Prove that every diagonal entry of a negative definite matrix is negative.

$\langle Av, v\rangle < 0$ for all nonzero $v \in \mathbb{R}^n$.

If $v = e_i$ (1 in the i^{th} coordinate, 0 in all other coordinates)

$\langle Ae_i, e_i\rangle < 0$

but Ae_i is the i^{th} column of i, so

$\langle Ae_i, e_i\rangle$ is the (i,i) entry of A. Since $1 \le i \le n$ is arbitrary, $\langle Ae_i, e_i\rangle < 0$ implies that every diagonal entry of A is negative.

Part (b)

Find, with proof, the distinct eigenvalues of a general lower triangular n by n matrix and their algebraic multiplicities.

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \text{char}_A(x) = \begin{vmatrix} a_{11}-x & 0 & \cdots & 0 \\ a_{21} & a_{22}-x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix}$$

$$= (a_{11}-x)(a_{22}-x)\cdots(a_{nn}-x)$$

So the eigenvalues (counted with algebraic multiplicity) are the diagonal entries of A,

$\lambda = a_{11}, a_{22}, \ldots, a_{nn}$.

- Orthogonally diagonalize the 4 by 4 checkerboard matrix

Problem 4 (30 points)

Show that $C = \{\overset{v_1}{(1,1,1,0)}, \overset{v_2}{(-2,1,1,0)}, \overset{v_3}{(0,-1,1,0)}, \overset{v_4}{(0,0,0,1)}\}$ is an orthogonal basis for $\mathbb{R}^4$.
Consider the linear transformation $T : P_3 \to \mathbb{R}^4$ given by

$$T(p(x)) = \begin{bmatrix} p(-10) \\ p(-1) \\ p(1) \\ p(10) \end{bmatrix}$$

Find $[T]_{\mathcal{B}\to\mathcal{C}}$ where $\mathcal{B} = \{1, x, x^2, x^3\}$ is the standard basis for P_3.

$(1,1,1,0)\cdot(-2,1,1,0) = 0 \qquad (1,1,1,0)\cdot(0,-1,1,0) = 0$

$(1,1,1,0)\cdot(0,0,0,1) = 0 \qquad (-2,1,1,0)\cdot(0,-1,1,0) = 0$

$(-2,1,1,0)\cdot(0,0,0,1) = 0 \qquad (0,-1,1,0)\cdot(0,0,0,1) = 0$

So C is an orthogonal set in $\mathbb{R}^4$. So C has linearly independent vectors and since C has four vectors and $\dim(\mathbb{R}^4) = 4$, C is an orthogonal basis for $\mathbb{R}^4$.

$T(1) = (1,1,1,1) = (1,1,1,0) + (0,0,0,1)$

$T(x) = (-10,-1,1,10)$

$c_1 = \dfrac{(-10,-1,1,10)\cdot(1,1,1,0)}{(1,1,1,0)\cdot(1,1,1,0)} = -\dfrac{10}{3} \qquad c_2 = \dfrac{(-10,-1,1,10)\cdot v_2}{v_2\cdot v_2} = \dfrac{20}{6} = \dfrac{10}{3}$

$c_3 = \dfrac{(-10,-1,1,10)\cdot(0,-1,1,0)}{(0,-1,1,0)\cdot(0,-1,1,0)} = 1 \qquad c_4 = \dfrac{(-10,-1,1,10)\cdot v_4}{v_4\cdot v_4} = 10$

$T(x^2) = (100,1,1,100) \qquad c_1 = \dfrac{(100,1,1,100)\cdot v_1}{v_1\cdot v_1} = \dfrac{102}{3} = 34$

$c_2 = \dfrac{(100,1,1,100)\cdot v_2}{v_2\cdot v_2} = \dfrac{-198}{6} = -33 \qquad c_3 = \dfrac{(100,1,1,100)\cdot v_3}{v_3\cdot v_3} = 0 \qquad c_4 = \dfrac{(100,1,1,100)\cdot v_4}{v_4\cdot v_4} = 100$

$T(x^3) = (-1000,-1,1,1000)$

$c_1 = \dfrac{(-1000,-1,1,1000)\cdot v_1}{v_1\cdot v_1} = -\dfrac{1000}{3} \qquad c_2 = \dfrac{(-1000,-1,1,1000)\cdot v_2}{v_2\cdot v_2} = \dfrac{2000}{6} = \dfrac{1000}{3}$

$c_3 = \dfrac{(-1000,-1,1,1000)\cdot v_3}{v_3\cdot v_3} = 1 \qquad c_4 = \dfrac{(-1000,-1,1,1000)\cdot v_4}{v_4\cdot v_4} = 1000$

$$[T]_{B\to C} = \begin{bmatrix} 1 & -\frac{10}{3} & 34 & -\frac{1000}{3} \\ 0 & \frac{10}{3} & -33 & \frac{1000}{3} \\ 0 & 1 & 0 & 1 \\ 1 & 10 & 100 & 1000 \end{bmatrix}$$

Problem 5 (20 points)

Find all least squares solutions to the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 2 & -3 & 4 & 0 \\ -2 & -2 & -4 & 6 & -8 & 0 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. What is the least squares error?

$A\hat{x} = \text{proj}_{\text{Col}(A)} b$

Every column of A is a multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

So an orthonormal basis for Col(A) is $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$.

$$\text{So } \text{proj}_{\text{Col}(A)} b = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \right\rangle \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$$

$$= \frac{2}{\sqrt{5}} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ -\frac{4}{5} \end{bmatrix}$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 2 & -3 & 4 & 0 & \frac{2}{5} \\ -2 & -2 & -4 & 6 & -8 & 0 & -\frac{4}{5} \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 2 & -3 & 4 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\hat{x} = \begin{pmatrix} \frac{2}{5} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$\| b - \text{proj}_{\text{Col}(A)} b \| = \| (2,0) - (\tfrac{2}{5}, -\tfrac{4}{5}) \|$$

$$= \| (\tfrac{8}{5}, \tfrac{4}{5}) \| = \boxed{\frac{4\sqrt{5}}{5}}$$ least squares error

Problem 6 (30 points)

Let $\{v_1, v_2\}$ be an orthonormal basis for $\mathbb{R}^2$. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator and that $\{Tv_1, Tv_2\}$ is also an orthonormal basis for $\mathbb{R}^2$. Prove that T is an isometry on $\mathbb{R}^2$.

First, show that T is bijective.

$$T(c_1 v_1 + c_2 v_2) = c_1 Tv_1 + c_2 Tv_2$$

- T is 1-1: If $c_1 Tv_1 + c_2 Tv_2 = 0$, then $c_1 = c_2 = 0$ since Tv_1, Tv_2 are linearly independent in $\mathbb{R}^2$. So $\ker(T) = \{0\}$.
- T is onto: Since $c_1 Tv_1 + c_2 Tv_2$ can attain any value in $\mathbb{R}^2$ (since Tv_1, Tv_2 span $\mathbb{R}^2$), T is onto.

So T is a bijective linear transformation.

Next, check that T preserves inner products.

$$\begin{aligned}
&\langle T(a_1 v_1 + a_2 v_2), T(b_1 v_1 + b_2 v_2)\rangle \\
&= \langle a_1 Tv_1 + a_2 Tv_2, b_1 Tv_1 + b_2 Tv_2\rangle \\
&= a_1 \langle Tv_1, b_1 Tv_1 + b_2 Tv_2\rangle + a_2 \langle Tv_2, b_1 Tv_1 + b_2 Tv_2\rangle \\
&= a_1 b_1 \underbrace{\langle Tv_1, Tv_1\rangle}_{1} + \cancel{a_1 b_2 \langle Tv_1, Tv_2\rangle} \\
&\qquad + \cancel{a_2 b_1 \langle Tv_2, Tv_1\rangle} + a_2 b_2 \underbrace{\langle Tv_2, Tv_2\rangle}_{1} \\
&= a_1 b_1 + a_2 b_2 \quad (\text{since } \{Tv_1, Tv_2\} \text{ is an ONB for } \mathbb{R}^2)
\end{aligned}$$

$$\begin{aligned}
&\langle a_1 v_1 + a_2 v_2, b_1 v_1 + b_2 v_2\rangle \\
&= a_1 \langle v_1, b_1 v_1 + b_2 v_2\rangle + a_2 \langle v_2, b_1 v_1 + b_2 v_2\rangle \\
&= a_1 b_1 \underbrace{\langle v_1, v_1\rangle}_{1} + \cancel{a_1 b_2 \langle v_1, v_2\rangle} + \cancel{a_2 b_1 \langle v_2, v_1\rangle} + a_2 b_2 \underbrace{\langle v_2, v_2\rangle}_{1} \\
&= a_1 b_1 + a_2 b_2 \quad (\text{since } \{v_1, v_2\} \text{ is an ONB for } \mathbb{R}^2)
\end{aligned}$$

So $\langle T(a_1 v_1 + a_2 v_2), T(b_1 v_1 + b_2 v_2)\rangle = \langle a_1 v_1 + a_2 v_2, b_1 v_1 + b_2 v_2\rangle$ and thus T is an isometry on $\mathbb{R}^2$.

Problem 7 (30 points)

Suppose that A is an arbitrary 2 by 2 matrix that has trace 1 and determinant -6.

Part (a)

Find the eigenvalues of A. [20 points]

Let λ_1 and λ_2 be the eigenvalues of A.

$tr(A) = 1 \Rightarrow \quad \lambda_1 + \lambda_2 = 1$

$det(A) = -6 \Rightarrow \quad \lambda_1 \lambda_2 = -6$

$\lambda_2 = 1 - \lambda_1 \qquad \lambda_1(1-\lambda_1) = -6$

$\lambda_1^2 - \lambda_1 - 6 = 0$

$(\lambda_1 - 3)(\lambda_1 + 2) = 0$

$\lambda_1 = 3$, $\lambda_2 = -2$ OR $\lambda_1 = -2$, $\lambda_2 = 3$

So the eigenvalues of A are -2 and 3.

Part (b)

Is A diagonalizable? Justify your answer. [10 points]

A is diagonalizable. Any eigenvalue has geometric multiplicity ≥ 1. So $\lambda = -2$ and $\lambda = 3$ both have multiplicity ≥ 1. So the sum of geometric multiplizities is ≥ 2. Since A is 2 by 2, this means that the sum of the geometric multiplicities is 2, so A is diagonalizable.

END OF EXAM