Math 54 Midterm 3 (Practice 3)

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SSID: Answer Key

Instructions:

- This exam is **120 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Consider the Bilaplacian $\Delta\Delta : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, given by $(\Delta\Delta)(f(x)) = f''''(x)$.

Part (a)

Show that every positive number is an eigenvalue. [15 points]

Given $k > 0$,

$\Delta\Delta(\sin(\sqrt[4]{k}\,x)) = k\sin(\sqrt[4]{k}\,x)$

where $\sqrt[4]{k}$ is defined since $k > 0$.

So every positive number is an eigenvalue of $\Delta\Delta$.

Part (b)

Is $\lambda = 0$ an eigenvalue of $\Delta\Delta$? If yes, find a basis for the eigenspace corresponding to $\lambda = 0$. [15 points]

$\Delta\Delta f = 0$

$f'''' = 0$

$\Rightarrow f = ax^3 + bx^2 + cx + d$

So 0 is an eigenvalue of $\Delta\Delta$. A basis for the eigenspace is $\boxed{y = 1, y = x, y = x^2, y = x^3}$.

Problem 2 (30 points)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & -2 \\ -2 & -2 & -1 \end{bmatrix}$$

Find the characteristic polynomial of A, all eigenvalues (and their algebraic and geometric multiplies), and all eigenvectors. If A is diagonalizable, diagonalize A. [20 points]

$$\begin{vmatrix} 1-x & 0 & 0 \\ -2 & -1-x & -2 \\ -2 & -2 & -1-x \end{vmatrix} = (1-x)(1+2x+x^2-4)$$
$$= (1-x)(x^2+2x-3)$$
$$= (1-x)(x-1)(x+3)$$
$$= \boxed{-(x+3)(x-1)^2 = \mathrm{char}_A(x)}$$

$\lambda = -3$ algebraic mult 1 $\qquad \lambda = 1$ algebraic mult 2

$$\begin{bmatrix} 4 & 0 & 0 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

diagonalizable $\quad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}^{-1}$

Part (b)

Calculate $(A^{-1})^{100}v$, where $v = (1, 1, 1)$. [10 points]

$$(1,1,1) = c_1(0,1,1) + c_2(1,-1,0) + c_3(1,0,-1)$$

$$\begin{aligned} c_2 + c_3 &= 1 \\ c_1 - c_2 &= 1 \\ c_1 - c_3 &= 1 \end{aligned} \qquad c_1 = \frac{3}{2} \quad c_2 = \frac{1}{2} \quad c_3 = \frac{1}{2}$$

$$(A^{-1})^{100}\left(\frac{3}{2}(0,1,1) + \frac{1}{2}(1,-1,0) + \frac{1}{2}(1,0,-1)\right)$$
$$= \left(-\frac{1}{3}\right)^{100} \frac{3}{2}(0,1,1) + \underbrace{\frac{1}{2}(1,-1,0) + \frac{1}{2}(1,0,-1)}_{\left(\frac{1}{1}\right)^{100}}$$

$$= \frac{1}{2 \cdot 3^{99}}(0,1,1) + \left(1, -\frac{1}{2}, -\frac{1}{2}\right)$$
$$= \boxed{\left(1, \frac{1}{2\cdot 3^{99}} - \frac{1}{2}, \frac{1}{2\cdot 3^{99}} - \frac{1}{2}\right)}$$

Problem 3 (30 points)

Short proofs. [15 points each]

Part (a)

Let V be an inner product space (not necessarily $\mathbb{R}^n$). Show that if v_1, v_2, and v_3 comprise an orthogonal set in V, then

$$||v_1 + v_2 + v_3||^2 = ||v_1||^2 + ||v_2||^2 + ||v_3||^2$$

$$\|v_1 + v_2 + v_3\|^2 = \langle v_1 + v_2 + v_3, v_1 + v_2 + v_3 \rangle$$
$$= \langle v_1, v_1 + v_2 + v_3 \rangle + \langle v_2, v_1 + v_2 + v_3 \rangle + \langle v_3, v_1 + v_2 + v_3 \rangle$$
$$= \langle v_1, v_1 \rangle + \cancel{\langle v_1, v_2 \rangle} + \cancel{\langle v_1, v_3 \rangle} + \cancel{\langle v_2, v_1 \rangle} + \langle v_2, v_2 \rangle$$
$$+ \cancel{\langle v_2, v_3 \rangle} + \cancel{\langle v_3, v_1 \rangle} + \cancel{\langle v_3, v_2 \rangle} + \langle v_3, v_3 \rangle$$
$$= \|v_1\|^2 + \|v_2\|^2 + \|v_3\|^2$$

Part (b)

Prove that for every positive definite matrix A, there exists a positive definite matrix B such that $B^4 = A$.

Since A is positive definite, it is symmetric and hence orthogonally diagonalizable as

$$A = ODO^T \text{ where } O^T = O^{-1}$$

and D is diagonal with positive diagonal entries.

Hence $\sqrt[4]{D}$ makes sense, as a diagonal matrix with the fourth root of each of the (positive) diagonal entries of D on the diagonal. Then,

$$B = O\sqrt[4]{D}\,O^T$$

satisfies $B^4 = A$ and since the diagonal entries of $\sqrt[4]{D}$ are still positive, B is positive definite.

Problem 4 (30 points)

Part (a)

Find the distance from the point $(2, 1, 1)$ to the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 0 & -1 \\ 2 & -2 & 2 \\ -1 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[20 points]

Basis for nullspace is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

So nullspace$(A) = \text{span}\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

$$\text{proj}_{\text{nullspace}(A)}(2,1,1) = \left\langle (2,1,1), \left(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right) \right\rangle \left(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right)$$

$$= \left(\tfrac{1}{2}, 0, -\tfrac{1}{2}\right)$$

$$\left\| (2,1,1) - \left(\tfrac{1}{2}, 0, -\tfrac{1}{2}\right) \right\| = \sqrt{\tfrac{9}{4} + 1 + \tfrac{9}{4}} = \boxed{\tfrac{\sqrt{22}}{2}}$$

Part (b)

Find the point in the nullspace of the matrix A (where A is the matrix in part (a)) that is closest to $(2, 1, 1)$.

$$\left(\tfrac{1}{2}, 0, -\tfrac{1}{2}\right)$$

Problem 5 (30 points)

Part (a)

Let A be a matrix with a nontrivial nullspace. Find an eigenvalue of A. [15 points]

$\boxed{\lambda = 0}$ since any nonzero vector in the eigenspace for $\lambda = 0$ is a nonzero vector in the nullspace of A and vice versa.

Part (b)

Let A be a matrix such that the entries in every row sum to a fixed number k. Find an eigenvalue of A. [15 points]

$\boxed{\lambda = k}$ since $A\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ k \\ \vdots \\ k \end{pmatrix} = k\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

Problem 6 (30 points)

Perform the Gram-Schmidt process on the vectors $\mathcal{B} = \{e^x, e^{2x}\}$ in $L^2([0,1])$. [30 points]

$v_1 = e^x$

$\langle v_1, v_1 \rangle = \int_0^1 e^x e^x dx$

$= \int_0^1 e^{2x} dx = \frac{1}{2} e^{2x} \Big|_0^1 = \frac{1}{2}(e^2 - 1)$

$$\boxed{w_1 = \frac{1}{\sqrt{\frac{1}{2}(e^2-1)}} e^x}$$

$v_2 = e^{2x}$

$v_2 - \langle v_2, w_1 \rangle w_1$

$= e^{2x} - \int_0^1 e^{2x} \frac{1}{\sqrt{\frac{1}{2}(e^2-1)}} e^x dx \frac{1}{\sqrt{\frac{1}{2}(e^2-1)}} e^x$

$= e^{2x} - \frac{1}{\frac{1}{2}(e^2-1)} e^x \left(\frac{1}{3}(e^3-1)\right)$

$= e^{2x} - \frac{2(e^3-1)}{3(e^2-1)} e^x$

$\int_0^1 \left(e^{2x} - \frac{2(e^3-1)}{3(e^2-1)} e^x\right)^2 dx$

$= \int_0^1 e^{4x} dx - \frac{4(e^3-1)}{3(e^2-1)} \int_0^1 e^{3x} dx + \frac{4(e^3-1)^2}{9(e^2-1)^2} \int_0^1 e^{2x} dx$

$= \frac{1}{4}(e^4-1) - \frac{4(e^3-1)}{3(e^2-1)} \frac{1}{3}(e^3-1) + \frac{4(e^3-1)^2}{9(e^2-1)^2} \frac{1}{2}(e^2-1)$

$w_2 = \left(\frac{1}{\frac{1}{4}(e^4-1) - \frac{4}{9}\frac{(e^3-1)^2}{e^2-1} + \frac{2}{9}\frac{(e^3-1)^2}{e^2-1}}\right)^{1/2} \left(e^{2x} - \frac{2}{3}\frac{e^3-1}{e^2-1} e^x\right)$

$$\boxed{w_2 = \left(\frac{1}{\frac{1}{4}(e^4-1) - \frac{2}{9}\frac{(e^3-1)^2}{e^2-1}}\right)^{1/2} \left(e^{2x} - \frac{2}{3}\frac{e^3-1}{e^2-1} e^x\right)}$$

Problem 7 (20 points)

Consider the matrices

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

First, determine if A is positive definite. Then, determine if B is positive semidefinite. [20 points]

Use Sylvester's criterion.

$|3| = 3$

$\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8$

$$\begin{vmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{vmatrix} = 3(8) - (-1)(-3)$$
$$= 27$$

So A is positive definite.

$$\begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{vmatrix} = (1-x)(\cancel{1} - 2x + x^2 \cancel{-1}) - 1(\cancel{1} - x \cancel{-1}) + 1(\cancel{1} \cancel{-1} + x)$$
$$= (1-x)(x^2 - 2x) + 2x$$
$$= -x^3 + x^2 + 2x^2 \cancel{-2x} \cancel{+2x}$$
$$= -x^3 + 3x^2 = x^2(3-x)$$

$$\lambda = 0, 0, 3$$

So B is positive semidefinite since all of its eigenvalues are nonnegative.

END OF EXAM