Math 54 Midterm 3 (Practice 2)

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SSID: Answer Key

Instructions:

- This exam is **120 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Is A positive definite? Is A positive semidefinite? What is the maximum value of $\frac{\langle Av,v\rangle}{\langle v,v\rangle}$ for a nonzero vector v?

$$\begin{vmatrix} 1-x & 0 & 1 \\ 0 & 3-x & 0 \\ 1 & 0 & 1-x \end{vmatrix} = (1-x)(3-x)(1-x) + 1(x-3)$$

$$= (3-x)((1-x)^2 - 1)$$

$$= (3-x)(-2x+x^2)$$

$$= x(x-2)(3-x)$$

$\lambda = 0, \lambda = 2, \lambda = 3$

$\boxed{\text{A is not positive definite, A is positive semidefinite.}}$

$$\boxed{\text{So } \max_{v \neq 0, v \in \mathbb{R}^n} \frac{\langle Av, v\rangle}{\langle v, v\rangle} = \lambda_{max} = 3}$$

$\lambda = 0$: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda = 2$: $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 3$: $\begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 2 (30 points)

Consider the following linear transformation $T : P_2 \to P_2$ given by

$$T(p(x)) = \frac{d}{dx}((x+1)p(x))$$

Part (a)

Find the determinant of T, the trace of T, all eigenvalues of T, and all eigenvectors of T. [15 points]

Use $\mathcal{B} = \{1, x, x^2\}$ $\quad T(1) = \frac{d}{dx}(x+1) = 1$

$T(x) = \frac{d}{dx}(x^2+x) = 2x+1$

$$[T]_{\mathcal{B}\to\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad T(x^2) = \frac{d}{dx}(x^3+x^2) = 3x^2+2x$$

$\det(T) = 1(6) = \boxed{6}$ $\quad tr(T) = 1+2+3 = \boxed{6}$

$$\begin{vmatrix} 1-x & 1 & 0 \\ 0 & 2-x & 2 \\ 0 & 0 & 3-x \end{vmatrix} = (1-x)(2-x)(3-x) \quad \lambda = 1,2,3$$

$$\lambda = 1: \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \lambda = 2 \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = 3 \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \lambda = 1 \quad y = 1 \quad \lambda = 2 \quad y = x+1 \\ \lambda = 3 \quad y = x^2 + 2x + 1 = (x+1)^2 \end{array}}$$

Part (b)

Find $T^{100}(x^2)$. [15 points]

$$x^2 = (x^2+2x+1) + (-2)(x+1) + 1(1)$$

$$T^{100}((x^2+2x+1) + (-2)(x+1) + (1)(1))$$

$$= 3^{100}(x^2+2x+1) + (-2)2^{100}(x+1) + 1$$

$$= \boxed{3^{100}x^2 + (2\cdot 3^{100} - 2\cdot 2^{100})x + (3^{100} - 2\cdot 2^{100} + 1)}$$

Problem 3 (30 points)

Short proofs. [15 points each]

Part (a)

Let V be an inner product space. Show that if $\langle v, u_1 \rangle = \langle v, u_2 \rangle$ for all $v \in V$, then $u_1 = u_2$.

$\langle v, u_1 \rangle - \langle v, u_2 \rangle = 0 \Rightarrow \langle v, u_1 - u_2 \rangle = 0$ for all $v \in V$.

So if $v = u_1 - u_2$, we get

$\langle u_1 - u_2, u_1 - u_2 \rangle = 0$

But by positive definiteness, $u_1 - u_2 = 0$.

So $u_1 = u_2$.

Part (b)

Let V be an inner product space, and let W be a subspace of V. Show that $W^\perp$ is a subspace of V, and that the only vector W in both W and $W^\perp$ is the zero vector.

Let $v_1, v_2 \in W^\perp$. We need to show that $c_1 v_1 + c_2 v_2 \in W^\perp$.

For any $w \in W$, $\langle w, v_1 \rangle = 0$, $\langle w, v_2 \rangle = 0$.

So $\langle w, c_1 v_1 + c_2 v_2 \rangle$

$= c_1 \langle w, v_1 \rangle + c_2 \langle w, v_2 \rangle$

$= c_1 \cdot 0 + c_2 \cdot 0 = 0$ for every $w \in W$.

Thus, $c_1 v_1 + c_2 v_2 \in W^\perp$ so $W^\perp$ is a subspace.

Suppose v is in both W and $W^\perp$. We claim that $v = 0$.

Since $v \in W^\perp$ it is perpendicular to every vector in W.
But since $v \in W$, it is hence perpendicular to itself.
So $\langle v, v \rangle = 0$. Thus, $v = 0$ by positive definiteness.

Problem 4 (30 points)

Show that if $T_1 : V \to V$ and $T_2 : V \to V$ are both isometries, then $T_2 \circ T_1 : V \to V$, defined by $(T_2 \circ T_1)(v) = T_2(T_1(v))$ is an isometry too.
[10 points]

If $T_1 : V \to V$, $T_2 : V \to V$ are both bijective linear operators, so is $(T_2 \circ T_1) : V \to V$.

$(T_2 \circ T_1)$ also preserves inner products.

$$\langle (T_2 \circ T_1) v_1, (T_2 \circ T_1) v_2 \rangle$$
$$= \langle T_2(T_1 v_1), T_2(T_1 v_2) \rangle$$
$$\underset{T_2 \text{ is an isometry}}{=} \langle T_1 v_1, T_1 v_2 \rangle \underset{T_1 \text{ is an isometry}}{=} \langle v_1, v_2 \rangle$$

So $(T_2 \circ T_1)$ is an isometry.

Part (b)

Find conditions on a, b, and c (if any exist) such that

$$T(x_1, x_2, x_3) = (ax_1 + x_2 + x_3, -x_1 + bx_2 + 2x_3, -2x_1 + cx_3)$$

is an isometry. [20 points]

$$T(x_1, x_2, x_3) = \begin{bmatrix} a & 1 & 1 \\ -1 & b & 2 \\ -2 & 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We need to find conditions on a, b, c so that

$$\begin{bmatrix} a & 1 & 1 \\ -1 & b & 2 \\ -2 & 0 & c \end{bmatrix} \text{ is an orthogonal matrix.}$$

But this can never be an orthogonal matrix since the columns can never be orthonormal since

$$a^2 + 1^2 + 2^2 \neq 1.$$

So no a, b, c exist.

Problem 5 (30 points)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} R_1 - R_2 \\ R_1 + R_3 \end{array}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Part (a)

Find an orthonormal basis for Col(A). [25 points]

Basis for Col(A): $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow w_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$v_2 - \langle v_2, w_1 \rangle w_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} - \overbrace{\frac{1}{\sqrt{3}} (-2) \frac{1}{\sqrt{3}}}^{\frac{2}{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

$$\frac{4}{9} + \frac{1}{9} + \frac{1}{9} + 1 = \frac{15}{9}$$

$$w_2 = \frac{3\sqrt{15}}{15} \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

$$\boxed{w_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad w_2 = \frac{1}{\sqrt{15}} \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix}}$$

Part (b)

Find the dimension of $(\text{Col}(A))^\perp$. [5 points]

$$\dim \text{Col}(A) + \dim(\text{Col}(A))^\perp = \dim \mathbb{R}^4.$$

So $\boxed{\dim(\text{Col}(A))^\perp = 4 - 2 = 2}$.

Problem 6 (30 points)

Find all a and b such that

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ a & b & 0 \end{bmatrix}$$

is diagonalizable.

$$\begin{vmatrix} 2-x & -2 & 0 \\ -2 & 2-x & 0 \\ a & b & -x \end{vmatrix} = (-x)(4-4x+x^2-4)$$

$$= (-x)(x^2-4x)$$

$$= -x^2(x-4)$$

$\lambda = 0$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ a & b & 0 \end{bmatrix}$$

needs geom mult $= 2$

So $\underline{a=-b}$.

$\lambda = 4$

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ a & b & -4 \end{bmatrix}$$

has geom mult $=1$

for every a, b. ✓

So need $a=-b$.

$\boxed{a=s,\ b=-s}$ for any $s \in \mathbb{R}$.

Problem 7 (20 points)

Part (a)

Find an example of a matrix that has no real eigenvalues, or show that it does not exist. [6 points]

need characteristic polynomial x^2+1

$$\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix} \Rightarrow \text{det is } x^2+1$$

$$\boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$

Part (b)

Find a matrix with all entries positive that is not positive definite, or show that it does not exist. [7 points]

$$\boxed{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}$$

not positive definite since it does not satisfy Sylvester's Criterion.

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \not> 0.$$

Part (c)

Find a matrix that has only one distinct eigenvalue but is not diagonalizable, or show that it does not exist. [7 points]

$$\boxed{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

$\lambda = 0$ has algebraic multiplicity 2, but geometric multiplicity 1.

END OF EXAM