Math 54 Midterm 3 (Practice 1)

Jeffrey Kuan

August 5, 2019

Name: Jeffrey Kuan

SSID: Answer Key

Instructions:

- This exam is **120 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Consider the following matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Find the geometric and algebraic multiplicity of every eigenvalue and a basis for each eigenspace. What is the characteristic polynomial of A? Is A diagonalizable? If so, diagonalize it.

This is upper triangular.

$$\mathrm{char}_A(x) = \begin{vmatrix} 2-x & 1 & 0 & 0 & 0 & 0 \\ 0 & 2-x & 0 & 0 & 0 & 0 \\ 0 & 0 & 3-x & 1 & 0 & 0 \\ 0 & 0 & 0 & 3-x & 1 & 0 \\ 0 & 0 & 0 & 0 & 3-x & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-x \end{vmatrix} = (2-x)^2(3-x)^3(-1-x)$$

$\lambda = -1, 2, 3$

algmult 1 algmult 2 algmult 3

$$\lambda=-1: A+I = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Basis: } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{geometric mult } 1$$

$$\lambda=2: A-2I = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix} \quad \text{Basis: } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{geometric mult } 1$$

$$\lambda=3: A-3I = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix} \quad \text{Basis: } \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{geometric mult } 1$$

A is not diagonalizable.

Problem 2 (20 points)

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix}$$

Find all least squares solutions to $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$. In addition, find the least squares error.

$$A^TA = \begin{bmatrix} 1 & 2 & -1 & -1 \\ 1 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -2 & 7 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 2 & -1 & -1 \\ 1 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$A^TA\hat{x} = A^Tb$$

$$\left[\begin{array}{cc|c} 7 & -2 & 3 \\ -2 & 7 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 7 & -2 & 3 \\ 1 & -\frac{7}{2} & -\frac{3}{2} \end{array}\right] -\tfrac{1}{2}R_2$$

$$\to \left[\begin{array}{cc|c} 0 & \frac{45}{2} & \frac{27}{2} \\ 1 & -\frac{7}{2} & -\frac{3}{2} \end{array}\right] R_1 - 7R_2 \to \left[\begin{array}{cc|c} 1 & -\frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{3}{5} \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{5} \\ 0 & 1 & \frac{3}{5} \end{array}\right]$$

$$-\frac{3}{2} + \frac{7}{2}\left(\frac{3}{5}\right) = -\frac{3}{2} + \frac{21}{10} = \frac{3}{5}$$

$$\boxed{\hat{x} = \left(\frac{3}{5}, \frac{3}{5}\right)}$$

$$A\hat{x} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \\ \frac{3}{5} \\ -\frac{6}{5} \end{bmatrix}$$

98
37
135 9·15

$$\|A\hat{x} - b\| = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{7}{5}\right)^2 + \left(\frac{7}{5}\right)^2 + \left(\frac{6}{5}\right)^2} = \boxed{\frac{3\sqrt{15}}{5}}$$

Problem 3 (30 points)

Short proofs. [15 points each]

Part (a)

Prove that if u and v are orthonormal vectors in an inner product space V (not necessarily $\mathbb{R}^n$) and a and b are real constants, then

$$||au + bv|| = \sqrt{a^2 + b^2}$$

$$||au + bv|| = \langle au + bv, au + bv \rangle^{1/2}$$
$$= (a\langle u, au + bv\rangle + b\langle v, au + bv\rangle)^{1/2}$$
$$= (a^2 \overset{1}{\langle u, u\rangle} + \cancel{ab\langle u, v\rangle} + \cancel{ab\langle v, u\rangle} + b^2 \overset{1}{\langle v, v\rangle})^{1/2}$$
$$= (a^2 + b^2)^{1/2} = \sqrt{a^2 + b^2} \checkmark$$

Part (b)

Show that if $v_1, v_2, \ldots, v_n$ form an orthonormal set in an inner product space V (not necessarily $\mathbb{R}^n$), then $v_1, v_2, \ldots, v_n$ are linearly independent.

Show that if $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$, then all $c_i = 0$.

$$\langle 0, v_i\rangle = \langle c_1v_1 + c_2v_2 + \ldots + c_nv_n, v_i\rangle$$
$$= c_1\langle v_1, v_i\rangle + c_2\langle v_2, v_i\rangle + \ldots + c_i\langle v_i, v_i\rangle + \ldots + c_n\langle v_n, v_i\rangle$$
$$= c_i\langle v_i, v_i\rangle = c_i \quad (\text{since } \langle v_j, v_i\rangle = 0 \text{ if } i \neq j, \ \langle v_i, v_i\rangle = 1)$$

So $c_i = 0$ for every i.

Thus, $c_i = 0$ for all i so $v_1, v_2, \ldots, v_n$ are linearly independent.

Problem 4 (30 points)

Define a **negative definite matrix** to be a symmetric n by n matrix A such that $\langle Av, v\rangle < 0$ for all nonzero vectors $v \in \mathbb{R}^n$.

Part (a)

Show that if A is a negative definite n by n matrix, then A has n negative eigenvalues. [15 points]

$\langle Av, v\rangle < 0$ for all nonzero vectors $v \in \mathbb{R}^n$.

Thus, $\dfrac{\langle Av, v\rangle}{\langle v, v\rangle} < 0$ for all nonzero vectors $v \in \mathbb{R}^n$ (since $\langle v, v\rangle > 0$)

Therefore, by Rayleigh's principle,

$$\max_{v \neq 0, v \in \mathbb{R}^n} \frac{\langle Av, v\rangle}{\langle v, v\rangle} = \lambda_{max} < 0$$

so every eigenvalue of A is negative.

Part (b)

Determine whether the matrix

$$A = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

is negative definite. (Hint: If A is negative definite, what is true about $-A$?) [15 points]

If $\langle Av, v\rangle < 0$, then $\langle -Av, v\rangle > 0$ for all nonzero $v \in \mathbb{R}^n$.
So A is negative definite if and only if $-A$ is positive definite.

$$-A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Use Sylvester's Criterion.

$|1| = 1 \qquad \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \leftarrow$ not positive

So $-A$ is not positive definite. Thus, A is not negative definite.

Problem 5 (30 points)

Part (a)

Find a unit vector for the vector $f(x) = x$ in $L^2([0,1])$. [5 points]

In $L^2([0,1])$, $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.

$$\langle x, x\rangle = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

So the unit vector is

$$\frac{x}{\left(\frac{1}{3}\right)^{1/2}} = \boxed{\sqrt{3}\,x}$$

Part (b)

Let W be the subspace of polynomials in P_2 that are orthogonal to $f(x) = x$ with respect to the $L^2([0,1])$ inner product. Find $\dim(W)$ and find a basis for W. [25 points]

Consider $a + bx + cx^2$

$$\int_0^1 x(a+bx+cx^2)\,dx$$

$$= \frac{a}{2}x^2 + \frac{b}{3}x^3 + \frac{c}{4}x^4\Big|_0^1$$

$$= \frac{a}{2} + \frac{b}{3} + \frac{c}{4}$$

Need $\frac{a}{2} + \frac{b}{3} + \frac{c}{4} = 0$

$$\Rightarrow 6a + 4a + 3c = 0$$

basis: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$

$$\boxed{\dim(W) = 2, \text{ basis is } 1 - 2x^2,\ 2 - 3x}$$

... the 4 by 4 checkerboard matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \cdots \end{bmatrix}$$

Problem 6 (30 points)

Let W consist of the set of points that lie on the intersection of the planes $x_1+x_2-x_3-2x_4=0$ and $x_1-2x_3+x_4=0$ in $\mathbb{R}^4$.

Part (a)

Find an orthonormal basis for W, ~~and an orthonormal basis for $W^\perp$~~. [20 points]

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 1 & 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 3 \end{bmatrix} R_2 - R_1$$

$$\to \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -3 \end{bmatrix} \quad \text{basis: } \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

Gram-Schmidt

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \boxed{w_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}}$$

$$v_2 - \langle v_2, w_1 \rangle w_1 = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{6}}(-5)\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{13}{6} \\ \frac{5}{6} \\ 1 \end{pmatrix}$$

$$\sqrt{\frac{4}{9} + \frac{169}{36} + \frac{25}{36} + 1} = \frac{\sqrt{246}}{6}$$

$$\begin{array}{r} 169 \\ 25 \\ 36 \\ 16 \\ \hline 246 \end{array}$$

$$\boxed{w_2 = \frac{6}{\sqrt{246}}\begin{pmatrix} \frac{2}{3} \\ \frac{13}{6} \\ \frac{5}{6} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{246}}\begin{pmatrix} 4 \\ 13 \\ 5 \\ 6 \end{pmatrix}}$$

Part (b)

Write $v = (20, 10, -10, -20)$ as a sum of vector in W and a vector in $W^\perp$. Is such a decomposition of v as a sum of vector in W and a vector in $W^\perp$ unique? [10 points]

Such a decomposition is unique since $\mathbb{R}^4 = W \oplus W^\perp$.

$$\langle v, w_1 \rangle = \frac{1}{\sqrt{6}}(40-10-10) = \frac{20}{\sqrt{6}} = \frac{10\sqrt{6}}{3}$$

$$\langle v, w_2 \rangle = \frac{1}{\sqrt{246}}(80+130-50-120) = \frac{40}{\sqrt{246}} = \frac{20\sqrt{246}}{123}$$

$$\frac{10\sqrt{6}}{3}\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{20}{3} \\ -\frac{10}{3} \\ \frac{10}{3} \\ 0 \end{pmatrix} \qquad \frac{20\sqrt{246}}{123}\frac{1}{\sqrt{246}}\begin{pmatrix} 4 \\ 13 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{80}{123} \\ \frac{260}{123} \\ \frac{100}{123} \\ \frac{120}{123} \end{pmatrix}$$

$$\begin{pmatrix} \frac{20}{3} \\ -\frac{10}{3} \\ \frac{10}{3} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{80}{123} \\ \frac{260}{123} \\ \frac{100}{123} \\ \frac{120}{123} \end{pmatrix} = \begin{pmatrix} \frac{900}{123} \\ \frac{-150}{123} \\ \frac{510}{123} \\ \frac{120}{123} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 20 \\ 10 \\ -10 \\ -20 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{900}{123} \\ -\frac{150}{123} \\ \frac{510}{123} \\ \frac{120}{123} \end{pmatrix}}_{\text{vector in } W} + \underbrace{\begin{pmatrix} \frac{1560}{123} \\ \frac{1380}{123} \\ -\frac{1740}{123} \\ -\frac{2580}{123} \end{pmatrix}}_{\text{vector in } W^\perp}}$$

Problem 7 (30 points)

Suppose that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ has $T(1,1,1) = 2$, $T(1,-1,0) = 1$, and $T(0,1,-1) = -1$. You may use the fact that $\{(1,1,1), (1,-1,0), (0,1,-1)\}$ is a basis for $\mathbb{R}^3$.

Part (a)

Calculate $T(4,26,29)$, $T(7,31,19)$, and T(-100, -20, 40). [20 points]

By the Riesz Representation Theorem, there is a unique vector w such that $T(v) = \langle v, w \rangle$. Let $w = (x_1, x_2, x_3)$.

$T(1,1,1) = 2 \qquad x_1 + x_2 + x_3 = 2$

$T(1,-1,0) = 1 \qquad x_1 - x_2 = 1$

$T(0,1,-1) = -1 \qquad x_2 - x_3 = -1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 3 & 3 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$(x_1, x_2, x_3) = (1, 0, 1)$

$T(4,26,29) = \langle (4,26,29), (1,0,1) \rangle = \boxed{33}$

$T(7,31,19) = \langle (7,31,19), (1,0,1) \rangle = 7 + 19 = \boxed{26}$

$T(-100,-20,40) = \langle (-100,-20,40), (1,0,1) \rangle = -100 + 40 = \boxed{-60}$

Part (b)

Find a basis for $(\ker(T))^\perp$. [10 points] (Hint: This should be quick if you did part (a) in an efficient way.)

$(\ker(T))^\perp$ is spanned by w since $T(v) = \langle v, w \rangle$.

So any vector $\perp$ to w is in $\ker(T)$.

So $\ker(T)$ = all vectors $\perp$ to w.

Hence $(\ker(T))^\perp = \text{span}\{w\}$.

$\boxed{\text{Basis for } (\ker(T))^\perp \text{ is } (1,0,1).}$

END OF EXAM