Math 54 Midterm 2 (Practice 4)

Jeffrey Kuan

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Name: Answer Key

SSID: ____________________

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **150 points**.
- Good luck!

Problem 1 (10 points)

Let V be the vector space of functions with period 2π (so that $f(x+2\pi) = f(x)$ for every x, so the function repeats itself after every 2π). For example, $\sin(x)$ and $\cos(x)$ are in V.

Part (a)

Define operations of vector addition and scalar multiplication on V. Check that V is closed under vector addition and scalar multiplication. What is the zero vector in V? [6 points]

vector add: $(f+g)(x) = f(x) + g(x)$

closed since sum of two periodic functions with period 2π still has period 2π.

scalar mult: $(cf)(x) = cf(x)$

closed since multiplying a function with period 2π by a real number still gives a function with period 2π.

zero vector is $f(x) = 0$, which clearly has period 2π.

Part (b)

Show that $T : \mathbb{R}^2 \to V$ defined by $T(a,b) = \underbrace{a\sin(x) + b\cos(x)}_{\in V}$ is a linear transformation. [4 points]

$$T(c_1(a_1,b_1) + c_2(a_2,b_2)) = T((c_1a_1 + c_2a_2,\ c_1b_1 + c_2b_2))$$
$$= (c_1a_1 + c_2a_2)\sin x + (c_1b_1 + c_2b_2)\cos x$$
$$= c_1(a_1\sin x + b_1\cos x) + c_2(a_2\sin x + b_2\cos x)$$
$$= c_1T(a_1,b_1) + c_2T(a_2,b_2) \checkmark$$

Problem 2 (15 points)

For each part, determine (with proof) if the set U is a subspace of the vector space V.

Part (a)

$$U = \text{the solutions to the}^{\text{in}}\text{homogeneous system } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$V = \mathbb{R}^3$$

not a subspace. solution set is the single point $(0,0,1)$
but this is not closed under scalar mult.,

$$\text{since } \underbrace{2(0,0,1)}_{\in U} = \underbrace{(0,0,2)}_{\notin U}.$$

Part (b)

U = the set of palindromes in $\mathbb{R}^4$, meaning points of the form (a, b, b, a) for real numbers a and b

$$V = \mathbb{R}^4$$

is a subspace.

$$c_1(a_1, b_1, b_1, a_1) + c_2(a_2, b_2, b_2, a_2)$$

$$= (\underbrace{c_1a_1 + c_2a_2}_{\text{"a"}}, \underbrace{c_1b_1 + c_2b_2}_{\text{"b"}}, \underbrace{c_1b_1 + c_2b_2}_{\text{"b"}}, \underbrace{c_1a_1 + c_2a_2}_{\text{"a"}})$$

Part (c)

$$U = \text{the set of continuous functions } f(x) \text{ such that } xf(x) = (f(x))^2$$

$$V = C(\mathbb{R})$$

(Hint: $f(x) = x$ is a function in U.)

not a subspace

$x \in U$ since $x(x) = (x)^2$

but $2x \notin U$ since $x(2x) \neq (2x)^2$

so U is not closed under scalar multiplication.

Problem 3 (30 points)

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations. If the linear transformation is bijective, find its inverse linear transformation.

Part (a)

The matrix transformation $T : M_{3\times 2} \to M_{3\times 2}$ given by

$$T(M) = AM$$

is a linear transformation

$$T(c_1M_1 + c_2M_2) = A(c_1M_1 + c_2M_2)$$
$$= c_1AM_1 + c_2AM_2$$
$$= c_1T(M_1) + c_2T(M_2) \checkmark$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. (Hint: What is $\det(A)$?)

$\det(A) = -1$ so A is invertible.

$\ker(T)$: $AM = 0$ $\quad A^{-1}AM = A^{-1}0$ $\quad M = 0$ $\quad \ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

$\text{range}(T) = M_{3\times 2}$ since given $B \in M_{3\times 2}$, $AM = B$ $\Rightarrow M = A^{-1}B$

so $T(A^{-1}B) = B$.

so bijective. $\boxed{T^{-1}(B) = A^{-1}B}$

(it's okay if you do not calculate A^{-1})

Part (b)

The norm map $N : \mathbb{C} \to \mathbb{R}$ given by

$$N(a + bi) = a^2 + b^2$$

not a linear transformation

$N(2 + 0i) = 4$ but

$N(2(2 + 0i)) = 16 \neq 2N(2 + 0i)$

Part (c)

The map $F : M_{2\times 2} \to \mathbb{R}^2$ given by

Note that this is also true!

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + b \\ c + d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a linear transformation $F(c_1M_1 + c_2M_2) = (c_1M_1 + c_2M_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= c_1M_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2M_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1F(M_1) + c_2F(M_2)$$

(or you can check this directly) $\checkmark$

$\ker(T)$: $\begin{matrix} a + b = 0 \\ c + d = 0 \end{matrix}$ $\quad \begin{array}{cccc} & \text{free} & & \text{free} \\ & \downarrow & & \downarrow \end{array}$ $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\quad (-s, s, -t, t) = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$$\boxed{\ker(T) = \left\{ s\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + t\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \right\}}$$

$\text{range}(T)$: $F\left(\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ c \end{bmatrix}$ so $\boxed{\text{range}(T) = \mathbb{R}^2}$ not one-to-one, so not bijective.

Problem 4 (25 points)

Consider the vector space P_1. This vector space has the following two bases.

$$\mathcal{B}_1 = \{1 + 3x, 2 - x\}$$

$$\mathcal{B}_2 = \{2 - 3x, 2 + 2x\}$$

Part (a)

Find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ and from $\mathcal{B}_2$ to $\mathcal{B}_1$. [10 points]

$1+3x = c_1(2-3x)+c_2(2+2x)$

$2c_1 + 2c_2 = 1 \qquad c_1 = -\frac{2}{5}$

$-3c_1 + 2c_2 = 3 \qquad c_2 = \frac{9}{10}$

$$[I]_{\mathcal{B}_1 \to \mathcal{B}_2} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{9}{10} & \frac{2}{5} \end{bmatrix}$$

$2-x = c_1(2-3x)+c_2(2+2x)$

$2c_1 + 2c_2 = 2$

$-3c_1 + 2c_2 = -1$

$c_1 = \frac{3}{5} \qquad c_2 = \frac{2}{5}$

$$[I]_{\mathcal{B}_2 \to \mathcal{B}_1} = \begin{bmatrix} -\frac{4}{7} & \frac{6}{7} \\ \frac{9}{7} & \frac{4}{7} \end{bmatrix}$$

$2-3x = c_1(1+3x)+c_2(2-x)$

$c_1 + 2c_2 = 2 \qquad c_1 = -\frac{4}{7}$

$3c_1 - c_2 = -3 \qquad c_2 = \frac{9}{7}$

$2+2x = c_1(1+3x)+c_2(2-x)$

$c_1 + 2c_2 = 2 \qquad c_1 = \frac{6}{7} \quad c_2 = \frac{4}{7}$

$3c_1 - c_2 = 2$

Part (b)

Let $\mathcal{C} = \{1, x, x^2, x^3\}$ denote the standard basis for P_3. Consider the linear transformation $\Delta : P_3 \to P_1$ given by $\Delta(p(x)) = p''(x)$. Find the matrix of Δ with respect to $\mathcal{C}$ and $\mathcal{B}_1$, $[\Delta]_{\mathcal{C}\to\mathcal{B}_1}$. [15 points]

$\Delta(1) = 0$

$\Delta(x) = 0$

$\Delta(x^2) = 2$

$\Delta(x^3) = 6x$

$2 = c_1(1+3x) + c_2(2-x)$

$c_1 + 2c_2 = 2 \qquad c_1 = \frac{2}{7} \quad c_2 = \frac{6}{7}$

$3c_1 - c_2 = 0$

$6x = c_1(1+3x)+c_2(2-x)$

$c_1 + 2c_2 = 0 \qquad c_1 = \frac{12}{7} \quad c_2 = -\frac{6}{7}$

$3c_1 - c_2 = 6$

$$[\Delta]_{\mathcal{C}\to\mathcal{B}_1} = \begin{bmatrix} 0 & 0 & \frac{2}{7} & \frac{12}{7} \\ 0 & 0 & \frac{6}{7} & -\frac{6}{7} \end{bmatrix}$$

Problem 5 (25 points)

Part (a)

Recall that $C(\mathbb{R})$ is the set of continuous functions defined on $\mathbb{R}$. Show that $T : C(\mathbb{R}) \to C(\mathbb{R})$ defined by

$$T(f) = \underbrace{(x-1)f(x)}_{\in C(\mathbb{R})}$$

is a linear transformation.

$$\begin{aligned} T(c_1 f_1 + c_2 f_2) &= (x-1)(c_1 f_1 + c_2 f_2) \\ &= c_1 (x-1) f_1(x) + c_2 (x-1) f_2(x) \\ &= c_1 T(f_1) + c_2 T(f_2) \checkmark \end{aligned}$$

Part (b)

Show that T is one-to-one.

Consider $\ker(T)$. $(x-1)f(x) = 0$

So if $x \neq 1$, then $x-1 \neq 0$ and hence $f(x) = 0$.

So $f(x) = 0$ for all $x \neq 1$.

But if f is a continuous function with $f(x) = 0$ for every $x \neq 1$, $f(x)$ must be identically 0. So $\ker(T) = \{0\}$ and hence T is one-to-one.

Part (c)

Show that T is not onto. (Hint: What is the value at $x = 1$ of $(x-1)f(x)$?)

Let g be any function in $C(\mathbb{R})$ with $g(1) \neq 0$, such as $f(x) = x^2$.

Then there is no $f \in C(\mathbb{R})$ such that

$$(x-1)f(x) = g(x)$$

since $(x-1)f(x)\big|_{x=1}$ is always 0, but $g(1) \neq 0$.

So $g \notin \text{range}(T)$ so T is not onto.

matrices

Problem 6 (25 points)

Suppose that W is a subspace of $M_{2\times 2}$ that has dimension 3.

Part (a)

Suppose you are told that the matrices $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, *and* $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ are in W. Show that these matrices form a basis for W. [15 points]

Show these matrices are linearly independent.

$$c_1 \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} 2c_1 + c_2 &= 0 \\ c_1 + c_2 + c_3 &= 0 \\ c_1 + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{array}{c} \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} R_1 - 2R_2 \\ R_2 - R_4 \\ R_3 - R_2 \\ \end{array}$$

$$\to \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 + R_4 \\ \\ -R_3 \\ R_3 + R_4 \end{array}$$

Since these three matrices are linearly ind. $\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $c_1 = c_2 = c_3 = c_4 = 0$.

in W, which has dim 3, they must span W too and hence they are a basis for W.

Part (b)

Find an example of a matrix in $M_{2\times 2}$ that is not in W.

By part (a),

$$W = \text{span}\left\{ \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

row red. does not change span of rows

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 1 & -2 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} R_1 - 2R_2 \to \begin{bmatrix} 0 & 0 & 2 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} R_1 + R_3$$

$$\to \begin{bmatrix} 0 & 0 & 1 & -\frac{1}{2} \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & \frac{3}{2} \end{bmatrix} R_3 - \tfrac{1}{2}R_1 \to \begin{bmatrix} 0 & 0 & 1 & -\frac{1}{2} \\ 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \end{bmatrix} R_2 - R_3$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

$$\text{So } W = \left\{ s\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & \frac{3}{2} \end{bmatrix} + u\begin{bmatrix} 0 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \right\}$$

So $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not in W (since
$$\begin{aligned} s &= 0 \\ t &= 0 \\ u &= 0 \\ -\tfrac{1}{2}s + \tfrac{3}{2}t - \tfrac{1}{2}u &= 1 \end{aligned}$$
has no solution)

Problem 7

Let S be the subspace of 4×4 matrices with trace equal to 0 such that all non-diagonal entries are zero.

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -a-b-c \end{bmatrix} \leftarrow \text{general matrix in } S.$$

Part (a)

Find a basis for S. Prove that the set you found is a basis. [20 points]

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -a-b-c \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

So take $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\}$

The equation above shows that this set spans S.
So we just need to check linear independence.

$$c_1\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow c_1 = 0 \quad c_2 = 0 \quad c_3 = 0 \quad -c_1 - c_2 - c_3 = 0 \Rightarrow$ only has trivial soln.

So $\mathcal{B}$ is linearly ind. and span S, so it is a basis for S.

Part (b)

Extend your basis from part (a) for the subspace S to a basis for all of $M_{4\times 4}$. You do not need to prove your answer. You can just state it. [5 points]

$$\left\{ \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}^{\mathcal{B}}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right\}$$

(Fourth matrix extends $\mathcal{B}$ to basis for all diagonal 4×4 matrices and the last eight non-diagonal matrices extend this to a basis for all 4×4 matrices)