Math 54 Midterm 2 (Practice 4)

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Name: ____

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Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **150 points**.
- Good luck!

Problem 1 (10 points)

Let V be the vector space of functions with period 2π (so that $f(x+2\pi) = f(x)$ for every x, so the function repeats itself after every 2π). For example, $\sin(x)$ and $\cos(x)$ are in V.

Part (a)

Define operations of vector addition and scalar multiplication on V. Check that V is closed under vector addition and scalar multiplication. What is the zero vector in V? [6 points]

Part (b)

Show that $T : \mathbb{R}^2 \to V$ defined by $T(a, b) = a\sin(x) + b\cos(x)$ is a linear transformation. [4 points]

Problem 2 (15 points)

For each part, determine (with proof) if the set U is a subspace of the vector space V.

Part (a)

$$U = \text{the solutions to the inhomogeneous system } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$V = \mathbb{R}^3$$

Part (b)

$U =$ the set of palindromes in $\mathbb{R}^4$, meaning points of the form (a, b, b, a) for real numbers a and b

$$V = \mathbb{R}^4$$

Part (c)

$$U = \text{the set of continuous functions } f(x) \text{ such that } xf(x) = (f(x))^2$$

$$V = C(\mathbb{R})$$

(Hint: $f(x) = x$ is a function in U.)

Problem 3 (30 points)

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations. If the linear transformation is bijective, find its inverse linear transformation.

Part (a)

The matrix transformation $T : M_{3\times 2} \to M_{3\times 2}$ given by

$$T(M) = AM$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. (Hint: What is $\det(A)$?)

Part (b)

The norm map $N : \mathbb{C} \to \mathbb{R}$ given by

$$N(a + bi) = a^2 + b^2$$

Part (c)

The map $F : M_{2\times 2} \to \mathbb{R}^2$ given by

$$F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + b \\ c + d \end{bmatrix}$$

Problem 4 (25 points)

Consider the vector space P_1. This vector space has the following two bases.

$$\mathcal{B}_1 = \{1 + 3x, 2 - x\}$$

$$\mathcal{B}_2 = \{2 - 3x, 2 + 2x\}$$

Part (a)

Find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ and from $\mathcal{B}_2$ to $\mathcal{B}_1$. [10 points]

Part (b)

Let $\mathcal{C} = \{1, x, x^2, x^3\}$ denote the standard basis for P_3. Consider the linear transformation $\Delta : P_3 \to P_1$ given by $\Delta(p(x)) = p''(x)$. Find the matrix of Δ with respect to $\mathcal{C}$ and $\mathcal{B}_1$, $[\Delta]_{\mathcal{C}\to\mathcal{B}_1}$. [15 points]

Problem 5 (25 points)

Part (a)

Recall that $C(\mathbb{R})$ is the set of continuous functions defined on $\mathbb{R}$. Show that $T : C(\mathbb{R}) \to C(\mathbb{R})$ defined by

$$T(f) = (x - 1)f(x)$$

is a linear transformation.

Part (b)

Show that T is one-to-one.

Part (c)

Show that T is not onto. (Hint: What is the value at $x = 1$ of $(x - 1)f(x)$?)

Problem 6 (25 points)

Suppose that W is a subspace of $M_{2\times 2}$ that has dimension 3.

Part (a)

Suppose you are told that the matrices $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, and \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ are in W. Show that these matrices form a basis for W. [15 points]

Part (b)

Find an example of a matrix in $M_{2\times 2}$ that is not in W.

Problem 7

Let S be the subspace of 4×4 matrices with trace equal to 0 such that all non-diagonal entries are zero.

Part (a)

Find a basis for S. Prove that the set you found is a basis. [20 points]

Part (b)

Extend your basis from part (a) for the subspace S to a basis for all of $M_{4\times 4}$. You do not need to prove your answer. You can just state it. [5 points]