Math 54 Midterm 2 (Practice 3)

Jeffrey Kuan

July 22, 2019

Name: Answer Key

SSID: ______________________

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **150 points**.
- Good luck!

Problem 1 (10 points)

Let $\mathbb{C}^2$ be the vector space of all ordered pairs, where each coordinate is a complex number. So a typical element of $\mathbb{C}^2$ would like like $(a+bi, c+di)$ where a, b, c, d are real numbers. As examples, $(1+i, 2-2i)$, $(1, 2+i)$, $(-i, 2)$ are elements of $\mathbb{C}^2$.

Part (a)

Define operations of vector addition and scalar multiplication on $\mathbb{C}^2$. Check that $\mathbb{C}^2$ is closed under vector addition and scalar multiplication. What is the zero vector in $\mathbb{C}^2$? [6 points]

vector add: $(a_1+b_1i, c_1+d_1i) + (a_2+b_2i, c_2+d_2i)$

$= ((a_1+a_2)+(b_1+b_2)i, (c_1+c_2)+(d_1+d_2)i)$

$\in \mathbb{C}^2$ so closed

scalar mult. $r(a+bi, c+di)$

$= (ar+bri, cr+dri)$

$\in \mathbb{C}^2$ so closed

zero vector: $(0+0i, 0+0i) = (0,0)$

Part (b)

What is the dimension of $\mathbb{C}^2$? Find (without proof) a basis for $\mathbb{C}^2$. [4 points]

$\dim(\mathbb{C}^2) = 4$

Basis: $B = \{(1,0), (i,0), (0,1), (0,i)\}$

Problem 2 (15 points)

For each part, determine (with proof) if the set U is a subspace of the vector space V.

Part (a)

$$U = \text{the set of matrices in } M_{5\times 5} \text{ whose trace is equal to } 1$$
$$V = M_{5\times 5}$$

not a subspace

not closed under scalar multiplication

$$\frac{1}{2}\underbrace{\begin{bmatrix} 1 & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ 0 & & & & 0 \end{bmatrix}}_{\in U} = \underbrace{\begin{bmatrix} \frac{1}{2} & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ 0 & & & & 0 \end{bmatrix}}_{\notin U}$$

Part (b)

$$U = \text{the set of continuous functions } f \text{ such that } f(2) \text{ is an integer}$$
$$V = C(\mathbb{R})$$

not a subspace

not closed under scalar mult.

$x^2 \in U$

but $\frac{1}{3}x^2 \notin U$ since $\frac{1}{3}(2)^2 = \frac{4}{3}$ is not an integer

Part (c)

$$U = \text{the set of elements in } \mathbb{C}^2 \text{ of the form } (a+bi, a-bi) \text{ where } a \text{ and } b \text{ are real numbers}$$
$$V = \mathbb{C}^2$$

(See Problem 1 for the definition of $\mathbb{C}^2$)

is a subspace

$$c_1(a_1+b_1 i, a_1-b_1 i) + c_2(a_2+b_2 i, a_2-b_2 i)$$
$$= (\underbrace{(c_1a_1+c_2a_2)}_{\text{"a"}} + \underbrace{(c_1b_1+c_2b_2)}_{\text{"b"}}i, \underbrace{(c_1a_1+c_2a_2)}_{\text{"a"}} - \underbrace{(c_1b_1+c_2b_2)}_{\text{"b"}}i)$$

Problem 3 (30 points)

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations. If the linear transformation is bijective, find its inverse linear transformation.

Part (a)

The map $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T\left(\overset{v}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}\right) = \overset{A}{\begin{bmatrix} 1 & 3 & -1 \\ -1 & -3 & 1 \\ 2 & 6 & -2 \\ 1 & 3 & -1 \end{bmatrix}} \overset{v}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

is a linear transf.

$T(c_1v_1 + c_2v_2)$
$= A(c_1v_1 + c_2v_2)$
$= c_1Av_1 + c_2Av_2$
$= c_1T(v_1) + c_2T(v_2)$ ✓

$\ker(T) = \text{nullspace}(A)$

$$\begin{bmatrix} 1 & 3 & -1 \\ -1 & -3 & 1 \\ 2 & 6 & -2 \\ 1 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (-3s+t, s, t) = \left\{ s\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

all columns are multiples of each other

$\text{range}(T) = \text{Col}(A) = \left\{ s\begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} \right\}$

not bijective since $\ker(T) \neq 0$ (not one-to-one)

Part (b)

The map $T : P_3 \to \mathbb{R}$ given by

$$T(p(x)) = p(-1) \cdot p(1)$$

not a linear transf.

$T(2p(x)) \neq 2T(p(x))$ since

$T(2p(x)) = (2p)(-1) \cdot (2p)(1) = 4p(-1)p(1)$
$= 4T(p(x))$

so $T(2p(x)) \neq 2T(p(x))$ for $p(x) = x^2$ for example.

Part (c)

The cyclic permutation map $P : M_{2\times3} \to M_{2\times3}$ given by

$$P\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} d & a & b \\ e & f & c \end{bmatrix}$$

is a linear transf. $P\left(r_1\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} + r_2\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix}\right)$

$= \begin{bmatrix} r_1d_1 + r_2d_2 & r_1a_1 + r_2a_2 & r_1b_1 + r_2b_2 \\ r_1e_1 + r_2e_2 & r_1f_1 + r_2f_2 & r_1c_1 + r_2c_2 \end{bmatrix} = r_1\begin{bmatrix} d_1 & a_1 & b_1 \\ e_1 & f_1 & c_1 \end{bmatrix} + r_2\begin{bmatrix} d_2 & a_2 & b_2 \\ e_2 & f_2 & c_2 \end{bmatrix}$

$= r_1P\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix}\right) + P\left(\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix}\right)$ ✓

$\ker(P) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$

$\text{range}(P) = M_{2\times3}$ since for any $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $P\left(\begin{bmatrix} b & c & f \\ a & d & e \end{bmatrix}\right) = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.

bijective, so $P^{-1}\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} b & c & f \\ a & d & e \end{bmatrix}$

Problem 4 (25 points)

Consider the vector space P_2. This vector space has the following two bases.

$$\mathcal{B}_1 = \{1, 2-x, 1+x^2\}$$

$$\mathcal{B}_2 = \{1+2x^2, -1-x^2, 1+x-x^2\}$$

Part (a)

Find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ and from $\mathcal{B}_2$ to $\mathcal{B}_1$. [10 points]

$$[I]_{\mathcal{B}_1\to\mathcal{B}_2} = \begin{bmatrix} -1 & -4 & 0 \\ -2 & -7 & -1 \\ 0 & -1 & 0 \end{bmatrix} \qquad [I]_{\mathcal{B}_2\to\mathcal{B}_1} = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

$1+2x^2 = 2(1+x^2) + (-1)(1)$

$-1-x^2 = (-1)(1+x^2)$

$1+x-x^2 = (-1)(1+x^2) + 2 + x$

$= (-1)(1+x^2) + (-1)(2-x) + 4(1)$

$(1+2x^2) + 2(-1-x^2) = -1$

$-1(1+2x^2) + (-2)(-1-x^2) = 1$

$2-x = c_1(1+2x^2) + c_2(-1-x^2) + c_3(1+x-x^2)$

$c_1 - c_2 + c_3 = 2 \quad c_3 = -1 \quad 2c_1 - c_2 - c_3 = 0$

$c_1 - c_2 = 3 \quad 2c_1 - c_2 = -1$

$c_1 = -4, c_2 = -7$

Part (b)

Consider the linear transformation $T : P_2 \to P_2$ given by $T(a + bx + cx^2) = b + cx + ax^2$. Find the matrix of T with respect to $\mathcal{B}_2$ and $\mathcal{B}_2$, $[T]_{\mathcal{B}_2\to\mathcal{B}_2}$. [15 points]

$T(1+0x+2x^2) = 0 + 2x + x^2$

$2x + x^2 = c_1(1+2x^2) + c_2(-1-x^2) + c_3(1+x-x^2)$

$c_1 - c_2 + c_3 = 0 \quad c_1 - c_2 = -2$

$c_3 = 2 \quad 2c_1 - c_2 = 3$

$2c_1 - c_2 - c_3 = 1 \quad c_1 = 5 \quad c_2 = 7$

$$[T]_{\mathcal{B}_2\to\mathcal{B}_2} = \begin{bmatrix} 5 & -3 & -2 \\ 7 & -4 & -4 \\ 2 & -1 & -1 \end{bmatrix}$$

$T(-1+0x-x^2) = 0 - x - x^2$

$-x-x^2 \quad c_1 - c_2 + c_3 = 0 \quad c_1 - c_2 = 1$

$c_3 = -1 \quad 2c_1 - c_2 = -2$

$2c_1 - c_2 - c_3 = -1 \quad c_1 = -3$

$c_2 = -4$

$T(1+x-x^2) = 1 - x + x^2$

$c_1 - c_2 + c_3 = 1 \quad c_1 - c_2 = 2$

$c_3 = -1 \quad 2c_1 - c_2 = 0$

$2c_1 - c_2 - c_3 = 1 \quad c_1 = -2 \quad c_2 = -4$

Problem 5 (25 points)

Part (a)

Show that the linear transformation $T : P_2 \to \mathbb{R}^3$ given by

$$T(p(x)) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$$

is one-to-one. [15 points]

$$T(a+bx+cx^2) = \begin{bmatrix} a-b+c \\ a \\ a+b+c \end{bmatrix}$$

ker(T): $a-b+c=0$, $a=0$, $a+b+c=0$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = (-1)(-2) = 2 \neq 0$$

so only soln is $a=b=c=0$.

So $\ker(T) = \{0+0x+0x^2\}$ so T is one-to-one.

Part (b)

Show that T is bijective. (Hint: You can do this easily using Rank-Nullity).

$\ker(T) = \{0+0x+0x^2\}$ so nullity$(T)=0$.

Since $\dim(P_2)=3$, rank$(T)=3$.

So range(T) is a subspace of $\mathbb{R}^3$ with dim 3 and hence range$(T)=\mathbb{R}^3$, so T is also onto.

So T is bijective.

Problem 6 (25 points)

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1,1,0,0) = (1,1,1)$$

$$T(2,1,0,0) = (0,1,1)$$

$$T(0,0,1,-1) = (1,2,2)$$

$$T(0,0,1,2) = (3,0,2)$$

Show that T is onto. From this, deduce using the Rank-Nullity Theorem that T is not bijective.

$(1,1,0,0), (2,1,0,0), (0,0,1,-1), (0,0,1,2)$ is a basis for $\mathbb{R}^4$.

since $c_1(1,1,0,0)+c_2(2,1,0,0)+c_3(0,0,1,-1)+c_4(0,0,1,2) = (0,0,0,0)$

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 + c_2 &= 0 \\ c_3 + c_4 &= 0 \\ -c_3 + 2c_4 &= 0 \end{aligned}$$

only has trivial soln

so there are 4 lin. ind. vectors in $\mathbb{R}^4$ and since $\dim(\mathbb{R}^4) = 4$, they must form a basis.

So $T(c_1(1,1,0,0) + c_2(2,1,0,0) + c_3(0,0,1,-1) + c_4(0,0,1,2))$
$= c_1(1,1,1) + c_2(0,1,1) + c_3(1,2,2) + c_4(3,0,2)$.

So range $(T) = \text{span}\{(1,1,1),(0,1,1),(1,2,2),(3,0,2)\}$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ R_3 - R_1 \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\text{span}\{(1,1,1),(0,1,1),(1,2,2),(3,0,2)\}$
$= \text{span}\{(1,0,0),(0,1,0),(0,0,1),(0,0,0)\} = \mathbb{R}^3$.

So range $(T) = \mathbb{R}^3$, hence T is onto.

So rank $(T) = 3$. Since $\dim(\mathbb{R}^4) = 4$, nullity $= \dim(\ker(T)) = 1$.

So $\ker(T) \neq \{0\}$.

So T is not one-to-one.

So T is not bijective.

Problem 7

Consider the set S of polynomials $p(x)$ in P_3 such that $p(1) = 0$ and $p'(1) = 0$. Show that S is a subspace of P_3, and find a basis for $\cancel{P_3}$ S and $\cancel{\dim(P_3)}$ $\dim(S)$.

$$a + bx + cx^2 + dx^3 = p(x)$$
$$b + 2cx + 3dx^2 = p'(x)$$

$$(*) \begin{cases} a + b + c + d = 0 \\ b + 2c + 3d = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{array}{cccc} & & \uparrow & \uparrow \\ & & \text{free} & \text{free} \end{array}$$

$$(s + 2t, -2s - 3t, s, t)$$
$$= s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \quad (**)$$

So take $\boxed{\mathcal{B} = \{1 - 2x + x^2, 2 - 3x + x^3\}}$.

The computation above shows $\mathcal{B}$ span S (since any $a + bx + cx^2 + dx^3$ in S satisfies $(*)$ and $(**)$ shows that such an $a + bx + cx^2 + dx^3$ is a linear combination of polynomials in $\mathcal{B}$).

linear independence: $c_1(1 - 2x + x^2) + c_2(2 - 3x + x^3) = 0$

$c_2 = 0 \quad c_1 = 0 \quad -2c_1 - 3c_2 = 0 \quad c_1 + 2c_2 = 0$

So $c_1 = c_2 = 0$. ✓

So $\mathcal{B}$ is a basis for S.

Thus, $\boxed{\dim(S) = 2}$

S is a subspace.

$p_1, p_2 \in S$.

Then $c_1 p_1 + c_2 p_2 \in S$ since

$$(c_1 p_1 + c_2 p_2)(1) = c_1 p_1(1) + c_2 p_2(1)$$
$$= c_1(0) + c_2(0) = 0$$

$$(c_1 p_1 + c_2 p_2)'(1) = c_1 p_1'(1) + c_2 p_2'(1)$$
$$= c_1(0) + c_2(0) = 0.$$