Math 54 Midterm 2 (Practice 3)

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July 22, 2019

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Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **150 points**.
- Good luck!

Problem 1 (10 points)

Let $\mathbb{C}^2$ be the vector space of all ordered pairs, where each coordinate is a complex number. So a typical element of $\mathbb{C}^2$ would like like $(a+bi, c+di)$ where a, b, c, d are real numbers. As examples, $(1+i, 2-2i)$, $(1, 2+i)$, $(-i, 2)$ are elements of $\mathbb{C}^2$.

Part (a)

Define operations of vector addition and scalar multiplication on $\mathbb{C}^2$. Check that $\mathbb{C}^2$ is closed under vector addition and scalar multiplication. What is the zero vector in $\mathbb{C}^2$? [6 points]

Part (b)

What is the dimension of $\mathbb{C}^2$? Find (without proof) a basis for $\mathbb{C}^2$. [4 points]

Problem 2 (15 points)

For each part, determine (with proof) if the set U is a subspace of the vector space V.

Part (a)

$$U = \text{the set of matrices in } M_{5\times 5} \text{ whose trace is equal to } 1$$
$$V = M_{5\times 5}$$

Part (b)

$$U = \text{the set of continuous functions } f \text{ such that } f(2) \text{ is an integer}$$
$$V = C(\mathbb{R})$$

Part (c)

$$U = \text{the set of elements in } \mathbb{C}^2 \text{ of the form } (a+bi, a-bi) \text{ where } a \text{ and } b \text{ are real numbers}$$
$$V = \mathbb{C}^2$$

(See Problem 1 for the definition of $\mathbb{C}^2$)

Problem 3 (30 points)

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations. If the linear transformation is bijective, find its inverse linear transformation.

Part (a)

The map $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & -1 \\ -1 & -3 & 1 \\ 2 & 6 & -2 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Part (b)

The map $T : P_3 \to \mathbb{R}$ given by

$$T(p(x)) = p(-1) \cdot p(1)$$

Part (c)

The cyclic permutation map $P : M_{2\times 3} \to M_{2\times 3}$ given by

$$P\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} d & a & b \\ e & f & c \end{bmatrix}$$

Problem 4 (25 points)

Consider the vector space P_2. This vector space has the following two bases.

$$\mathcal{B}_1 = \left\{1, 2 - x, 1 + x^2\right\}$$

$$\mathcal{B}_2 = \left\{1 + 2x^2, -1 - x^2, 1 + x - x^2\right\}$$

Part (a)

Find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ and from $\mathcal{B}_2$ to $\mathcal{B}_1$. [10 points]

Part (b)

Consider the linear transformation $T : P_2 \to P_2$ given by $T(a + bx + cx^2) = b + cx + ax^2$. Find the matrix of T with respect to $\mathcal{B}_2$ and $\mathcal{B}_2$, $[T]_{\mathcal{B}_2 \to \mathcal{B}_2}$. [15 points]

Problem 5 (25 points)

Part (a)

Show that the linear transformation $T : P_2 \to \mathbb{R}^3$ given by

$$T(p(x)) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$$

is one-to-one. [15 points]

Part (b)

Show that T is bijective. (Hint: You can do this easily using Rank-Nullity).

Problem 6 (25 points)

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1,1,0,0) = (1,1,1)$$

$$T(2,1,0,0) = (0,1,1)$$

$$T(0,0,1,-1) = (1,2,2)$$

$$T(0,0,1,2) = (3,0,2)$$

Show that T is onto. From this, deduce using the Rank-Nullity Theorem that T is not bijective.

Problem 7

Consider the set S of polynomials $p(x)$ in P_3 such that $p(1) = 0$ and $p'(1) = 0$. Show that S is a subspace of P_3, and find a basis for S and $\dim(S)$.