Math 54 Midterm 2 (Practice 2)

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Name: Answer Key

SSID:

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **150 points**.
- Good luck!

Problem 1 (10 points)

Let V of all sequences that are eventually zero, meaning that for some N sufficiently large, $a_n = 0$ for all $n \geq N$.

Part (a)

Define operations of vector addition and scalar multiplication on V. Check that V is closed under vector addition and scalar multiplication. What is the zero vector in V? [6 points]

vector addition: add sequences componentwise

$$\begin{array}{r} a_1, a_2, \ldots, 0, 0, 0, 0 \ldots \\ + \; b_1, b_2, \ldots, 0, 0, 0, 0, 0, 0 \\ \hline a_1+b_1, a_2+b_2, \ldots, 0, 0, 0, 0, 0 \end{array}$$ still eventually zero

scalar mult: multiply each entry by scalar

$$c(a_1, a_2, \ldots, 0, 0, 0, \ldots) = (ca_1, ca_2, \ldots, 0, 0, 0, \ldots)$$

still eventually zero

zero vector is the sequence of all zeros, which is clearly eventually zero.

$$0, 0, 0, \ldots$$

Part (b)

Consider the deletion linear transformation $T : V \to V$ given by

$$T(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots)$$

Find $\ker(T)$ and $\text{range}(T)$. [4 points]

$$\ker(T) = \{(a, 0, 0, 0, \ldots)\}$$

a any number

$\text{range}(T) = V$ since given $(a_1, a_2, a_3, \ldots, 0, 0, 0, \ldots)$,

$$T((1, a_1, a_2, a_3, \ldots, 0, 0, 0, \ldots))$$
$$= (a_1, a_2, a_3, \ldots, 0, 0, 0, \ldots)$$

Problem 2 (15 points)

For each part, determine (with proof) if the set U is a subspace of the vector space V.

Part (a)

$$U = \text{the set of matrices in } M_{4\times 4} \text{ whose diagonal entries are all } 0$$
$$V = M_{4\times 4}$$

is a subspace

$$c_1\begin{pmatrix} 0 & & * \\ & 0 & \\ * & & 0 & \\ & & & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & & * \\ & 0 & \\ * & & 0 & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & & * \\ & 0 & \\ * & & 0 & \\ & & & 0 \end{pmatrix}$$

Part (b)

$$U = \text{the set of } a + bi \text{ where } a \text{ and } b \text{ are integers}$$
$$V = \mathbb{C}$$

not a subspace

not closed under scalar multiplication

$$\frac{1}{2}\underbrace{(1+i)}_{\in U} = \underbrace{\frac{1}{2} + \frac{1}{2}i}_{\notin U}$$

Part (c)

$$U = \text{the set of functions } f \text{ in } C^\infty(\mathbb{R}) \text{ that satisfy } f' + 2f = 0$$
$$V = C^\infty(\mathbb{R})$$

(Recall that $C^\infty(\mathbb{R})$ is the vector space of smooth functions, which are functions that have infinitely many derivatives).

is a subspace.

If $f, g \in U$, then $c_1 f + c_2 g \in U$ too since then,

$$(c_1 f + c_2 g)' + 2(c_1 f + c_2 g)$$
$$= c_1 f' + c_2 g' + c_1 2f + c_2 2g$$
$$= c_1(f' + 2f) + c_2(g' + 2g)$$
$$= c_1(0) + c_2(0) = 0 \checkmark$$

Problem 3 (25 points)

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations. If the linear transformation is bijective, find its inverse linear transformation.

Part (a)

The map $F : M_{4\times4}$ given by $F(M) = AMA$ where A is an invertible 4 by 4 matrix.

is a linear transformation

$$F(c_1M_1+c_2M_2) = A(c_1M_1+c_2M_2)A = c_1AM_1A + c_2AM_2A$$
$$= c_1F(M_1) + c_2F(M_2)$$

$\ker(F) = \{$ zero matrix in $M_{4\times4}\}$

$$AMA = 0 \quad A^{-1}AMAA^{-1} = A^{-1}0A^{-1} \Rightarrow M = 0$$

$\text{range}(F) = M_{4\times4}$

If B is any matrix, solve $AMA = B$ for M. $M = A^{-1}BA^{-1}$.

so onto, one-to-one, bijective. $T^{-1}(B) = A^{-1}BA^{-1}$.

Part (b)

The map $T : P_3 \to \mathbb{R}^2$ given by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p'(0) \end{bmatrix}$$

is a linear transformation

$$T(c_1p_1+c_2p_2) = \begin{bmatrix} (c_1p_1+c_2p_2)(0) \\ (c_1p_1+c_2p_2)'(0) \end{bmatrix} = \begin{bmatrix} c_1p_1(0)+c_2p_2(0) \\ c_1p_1'(0)+c_2p_2'(0) \end{bmatrix}$$
$$= c_1\begin{bmatrix} p_1(0) \\ p_1'(0) \end{bmatrix} + c_2\begin{bmatrix} p_2(0) \\ p_2'(0) \end{bmatrix} = c_1T(p_1) + c_2T(p_2)$$

kernel: $a + bx + cx^2 + dx^3$

$p(0) = 0 \quad p'(0) \qquad a = 0$

$p'(x) = b + 2cx + 3dx^2 \Rightarrow p'(0) = b = 0$

so $\ker(T) = \{cx^2 + dx^3$ where c, d real$\}$

$\text{range}(T) = \mathbb{R}^2$ since $T(a+bx+cx^2+dx^3) = \begin{bmatrix} a \\ b \end{bmatrix}$.

Problem 4 (25 points)

Consider the vector space $\mathrm{Sym}_{2\times 2}$ of symmetric 2 by 2 matrices, which has the following bases.

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \right\}$$

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \right\}$$

Part (a)

Find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. [10 points]

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

$c_1 = 1 \quad 2c_2 - c_3 = 1, \; -2c_1 + c_2 + 2c_3 = 1 \Rightarrow c_2 = 1 \quad c_3 = 1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

$c_1 = 1 \quad 2c_2 - c_3 = 0, \; -2c_1 + c_2 + 2c_3 = 1 \Rightarrow c_2 = \frac{3}{5} \quad c_3 = \frac{6}{5}$

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

$c_1 = 1 \quad 2c_2 - c_3 = -1, \; -2c_1 + c_2 + 2c_3 = -1 \quad c_2 = -\frac{1}{5} \quad c_3 = \frac{3}{5}$

$$[I]_{\mathcal{B}_1 \to \mathcal{B}_2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{3}{5} & -\frac{1}{5} \\ 1 & \frac{6}{5} & \frac{3}{5} \end{bmatrix}$$

Part (b)

Consider the linear transformation $T : \mathrm{Sym}_{2\times 2} \to \mathrm{Sym}_{2\times 2}$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$.

Find the matrix of T with respect to $\mathcal{B}_2$ and $\mathcal{B}_2$, $[T]_{\mathcal{B}_2 \to \mathcal{B}_2}$. [15 points]

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

$c_1 = -2 \quad 2c_2 - c_3 = 0, \; -2c_1 + c_2 + 2c_3 = 1$

$c_2 = -3/5 \quad c_3 = -6/5$

$$T\left(\begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

$c_1 = 1 \quad 2c_2 - c_3 = 2, \; -2c_1 + c_2 + 2c_3 = 0 \quad c_2 = \frac{6}{5} \quad c_3 = \frac{2}{5}$

$$T\left(\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

$c_1 = 2 \quad 2c_2 - c_3 = -1, \; -2c_1 + c_2 + 2c_3 = 0$

$c_2 = \frac{2}{5} \quad c_3 = \frac{9}{5}$

$$[T]_{\mathcal{B}_2 \to \mathcal{B}_2} = \begin{bmatrix} -2 & 1 & 2 \\ -3/5 & \frac{6}{5} & \frac{2}{5} \\ -6/5 & \frac{2}{5} & \frac{9}{5} \end{bmatrix}$$

Problem 5 (25 points)

Part (a)

Find a basis for the subspace S_3 of polynomials in P_3 that have a zero at $x = 1$. Prove that the set you find is a basis. [15 points]

Find $\dim(S_3)$. Define $T: P_3 \to \mathbb{R}$ by $T(p(x)) = p(1)$ and note that range $(T) = \mathbb{R}$ (since $x+a-1 \to a$) so rank$(T) = 1$, nullity$(T) = 3$. So $\dim S_3 = 3$ since $S_3 = \ker T$.

So let $\boxed{B = \{(x-1), (x-1)^2, (x-1)^3\} = \{x-1, x^2-2x+1, x^3-3x^2+3x-1\}.}$

Check linear independence:

$$c_1(x-1) + c_2(x^2-2x+1) + c_3(x^3-3x^2+3x-1) = 0$$

$$c_3 = 0$$
$$c_2 - 3c_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0 \checkmark$$
$$c_1 - 2c_2 + 3c_3 = 0$$
$$-c_1 + c_2 - c_3 = 0$$

Since the vectors in B are linearly independent, they are three linearly ind. vectors in S_3 which has $\dim(S_3) = 3$, so they must also span and hence are a basis.

Part (b)

Now let S_{100} be the subspace of polynomials in P_{100} that have a zero at $x = 1$. Find $\dim(S_{100})$, and rigorously justify your answer. [10 points]

$T: P_{100} \to \mathbb{R}$ is a linear transf.

$f(x) \to f(1)$

range $(T) = \mathbb{R}$ $(x+a-1 \to a)$

so rank $(T) = 1$

nullity $(T) = 100$.

Since $S_{100} = \ker(T)$, $\dim(S_{100}) = 100$.

Problem 6 (25 points)

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1,1,0) = (1,0,1)$$
$$T(2,1,0) = (0,1,1)$$
$$T(1,1,1) = (-2,2,0)$$

Show that T is not one-to-one. (Hint: $\{(1,1,0),(2,1,0),(1,1,1)\}$ is an ordered basis for $\mathbb{R}^3$. If you want to use this fact, however, you must prove it.)

$(1,1,0),(2,1,0),(1,1,1)$ is an ordered basis for $\mathbb{R}^3$.

Lin. Ind: $c_1(1,1,0)+c_2(2,1,0)+c_3(1,1,1)=(0,0,0)$

$$c_1+2c_2+c_3=0 \qquad \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0.$$
$$c_1+c_2+c_3=0$$
$$c_3=0$$

So $c_1=c_2=c_3=0$

So there are three lin. ind. vectors in $\mathbb{R}^3$, which has dim 3, so they must also span and hence are a basis.

So every vector in $\mathbb{R}^3$ can be expressed uniquely as
$$c_1(1,1,0)+c_2(2,1,0)+c_3(1,1,1).$$

$$T(c_1(1,1,0)+c_2(2,1,0)+c_3(1,1,1))$$
$$=c_1(1,0,1)+c_2(0,1,1)+c_3(-2,2,0)$$

We need to show $\ker(T) \neq \{(0,0,0)\}$. So we must show
$$c_1(1,0,1)+c_2(0,1,1)+c_3(-2,2,0)=(0,0,0)$$
$$\begin{cases} c_1-2c_3=0 \\ c_2+2c_3=0 \\ c_1+c_2=0 \end{cases}$$ has a nontrivial soln.

But this is clear since $\begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix} = 1(-2)+(-2)(-1)=0.$

Problem 7 (Miscellanea: 25 points)

Part (a)

Show that the functions $y = 1$, $y = 2^x$, and $y = 2^{-x}$ are linearly independent as vectors in $C(\mathbb{R})$. [15 points]

$$c_1(1) + c_2(2^x) + c_3(2^{-x}) = 0.$$

$$\begin{aligned} x = 0: &\quad c_1 + c_2 + c_3 = 0 \\ x = 1: &\quad c_1 + 2c_2 + \tfrac{1}{2}c_3 = 0 \\ x = -1: &\quad c_1 + \tfrac{1}{2}c_2 + 2c_3 = 0 \end{aligned}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & \frac{1}{2} \\ 1 & \frac{1}{2} & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} = \frac{3}{4} \neq 0$$

so only solution is $c_1 = c_2 = c_3 = 0$.

Part (b)

Let $T : P_4 \to \mathbb{R}$ be a nonzero linear transformation (so there is some polynomial $p(x)$ of degree ≤ 4 such that $T(p(x))$ is nonzero). Find dim(ker(T)). [10 points]

(Hint: Rank-Nullity Theorem)

Since T is nonzero lin. transf., for some $a \neq 0$, $a \in$ range(T) which is a subspace of $\mathbb{R}$.

But any subspace of $\mathbb{R}$ containing a nonzero number must be all of $\mathbb{R}$. So range$(T) = \mathbb{R}$.

So rank$(T) = 1$, hence nullity$(T) = 4$

so $\boxed{\dim(\ker(T)) = 4}$.