Math 54 Midterm 2 (Practice 2)

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Name: ____

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Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **150 points**.
- Good luck!

Problem 1 (10 points)

Let V of all sequences that are eventually zero, meaning that for some N sufficiently large, $a_n = 0$ for all $n \geq N$.

Part (a)

Define operations of vector addition and scalar multiplication on V. Check that V is closed under vector addition and scalar multiplication. What is the zero vector in V? [6 points]

Part (b)

Consider the deletion linear transformation $T : V \to V$ given by

$$T(a_0, a_1, a_2, ...) = (a_1, a_2, a_3, ...)$$

Find $\ker(T)$ and $\text{range}(T)$. [4 points]

Problem 2 (15 points)

For each part, determine (with proof) if the set U is a subspace of the vector space V.

Part (a)

$$U = \text{the set of matrices in } M_{4\times 4} \text{ whose diagonal entries are all } 0$$
$$V = M_{4\times 4}$$

Part (b)

$$U = \text{the set of } a + bi \text{ where } a \text{ and } b \text{ are integers}$$
$$V = \mathbb{C}$$

Part (c)

$$U = \text{the set of functions } f \text{ in } C^\infty(\mathbb{R}) \text{ that satisfy } f' + 2f = 0$$
$$V = C^\infty(\mathbb{R})$$

(Recall that $C^\infty(\mathbb{R})$ is the vector space of smooth functions, which are functions that have infinitely many derivatives).

Problem 3 (25 points)

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations. If the linear transformation is bijective, find its inverse linear transformation.

Part (a)

The map $F : M_{4\times4}$ given by $F(M) = AMA$ where A is an invertible 4 by 4 matrix.

Part (b)

The map $T : P_3 \to \mathbb{R}^2$ given by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p'(0) \end{bmatrix}$$

Problem 4 (25 points)

Consider the vector space $\mathrm{Sym}_{2\times 2}$ of symmetric 2 by 2 matrices, which has the following bases.

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \right\}$$

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \right\}$$

Part (a)

Find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. [10 points]

Part (b)

Consider the linear transformation $T : \mathrm{Sym}_{2\times 2} \to \mathrm{Sym}_{2\times 2}$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$. Find the matrix of T with respect to $\mathcal{B}_2$ and $\mathcal{B}_2$, $[T]_{\mathcal{B}_2 \to \mathcal{B}_2}$. [15 points]

Problem 5 (25 points)

Part (a)

Find a basis for the subspace S_3 of polynomials in P_3 that have a zero at $x = 1$. Prove that the set you find is a basis. [15 points]

Part (b)

Now let S_{100} be the subspace of polynomials in P_{100} that have a zero at $x = 1$. Find $\dim(S_{100})$, and rigorously justify your answer. [10 points]

Problem 6 (25 points)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1,1,0) = (1,0,1)$$

$$T(2,1,0) = (0,1,1)$$

$$T(1,1,1) = (-2,2,0)$$

Show that T is not one-to-one. (Hint: $\{(1,1,0),(2,1,0),(1,1,1)\}$ is an ordered basis for $\mathbb{R}^3$. If you want to use this fact, however, you must prove it.)

Problem 7 (Miscellanea: 25 points)

Part (a)

Show that the functions $y = 1$, $y = 2^x$, and $y = 2^{-x}$ are linearly independent as vectors in $C(\mathbb{R})$. [15 points]

Part (b)

Let $T : P_4 \to \mathbb{R}$ be a nonzero linear transformation (so there is some polynomial $p(x)$ of degree ≤ 4 such that $T(p(x))$ is nonzero). Find $\dim(\ker(T))$. [10 points]

(Hint: Rank-Nullity Theorem)