Math 54 Midterm 2 (Practice Exam 1)

Jeffrey Kuan

July 22, 2019

Name: Answer Key

SSID:

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **150 points**.
- Good luck!

Problem 1 (10 points)

Let $D(\mathbb{R})$ be the set of continuous functions on $\mathbb{R}$ that decay rapidly. In particular, we say that a function f is in $D(\mathbb{R})$ if and only if both of the following statements are true:

$$\lim_{x\to\infty} f(x) = 0 \text{ and } \lim_{x\to-\infty} f(x) = 0$$

This is a vector space over $\mathbb{R}$.

Part (a)

Give three examples of nonzero functions in $D(\mathbb{R})$. [3 points]

$f(x) = \frac{1}{x^2+1}$, $f(x) = \frac{2}{x^2+1}$, $f(x) = \frac{3}{x^2+1}$

Part (b)

Show that $D(\mathbb{R})$ is closed under vector addition and scalar multiplication. What is the zero vector in $D(\mathbb{R})$? [7 points]

vector addition: $(f+g)(x) = f(x) + g(x)$

If $f, g \in D(\mathbb{R})$, $f+g$ is continuous (since f, g are)

and $\lim_{x\to\infty} (f+g)(x) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = 0+0=0$

$\lim_{x\to-\infty} (f+g)(x) = \lim_{x\to-\infty} f(x) + \lim_{x\to-\infty} g(x) = 0+0=0$.

So $f+g \in D(\mathbb{R})$

scalar multiplication: $(cf)(x) = cf(x)$

If $f \in D(\mathbb{R})$, cf is continuous and

$\lim_{x\to\infty} cf(x) = c\lim_{x\to\infty} f(x) = c \cdot 0 = 0$

$\lim_{x\to-\infty} cf(x) = c\lim_{x\to-\infty} f(x) = c \cdot 0 = 0$

so $cf \in D(\mathbb{R})$.

Zero vector is $f(x)=0$. Indeed, $f(x)=0$ is continuous and $\lim_{x\to\infty} 0 = 0$, $\lim_{x\to-\infty} 0 = 0$.

Problem 2 (20 points)

For each part, determine (with proof) if the set U is a subspace of the vector space V. If yes, find the dimension of U.

Part (a)

$U =$ the set of $a + bi$ where a and b are real numbers with $a \geq b$

$$V = \mathbb{C}$$

not a subspace

not closed under scalar multiplication.

$$-1\underbrace{(2+i)}_{\in U} = \underbrace{-2+(-1)i}_{\notin U}$$

Part (b)

$U =$ the set of polynomials in P_4 that have a zero at $x = 2$

$$V = P_4$$

is a subspace If $f, g \in U$, $c_1 f + c_2 g \in U$ too

since $(c_1 f + c_2 g)(2) = c_1 f(2) + c_2 g(2)$

$= c_1(0) + c_2(0) = 0.$

To calculate dimension, consider $T: P_4 \to \mathbb{R}$ defined by

$T(f) = f(2)$. Range is $\mathbb{R}$ since

$T: a + (x-2) \longmapsto a$. So $\text{rank}(T) = 1$.

So $\text{nullity}(T) = 4$ $(\dim(P_4) = 5)$ and

since $\ker(T) = U$,

$\boxed{\dim(U) = 4}$

Part (c)

$U =$ the set of polynomials in P_4 such that $f(0) = 2$

$$V = P_4$$

not a subspace

$f(x) = x^2 + 2 \in U$ but $2f(x) = 2x^2 + 4 \notin U$

Problem 3 (20 points)

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations. If the linear transformation is bijective, find its inverse linear transformation.

Part (a)

The squaring map $T : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T(f) = f^2$.

not a linear transformation

$T(2f) \neq 2T(f)$ since

$$T(2f) = (2f)^2 = 4f^2 = 4T(f).$$

Part (b)

The conjugation map $T : M_{3\times3} \to M_{3\times3}$ given by $T(M) = A^{-1}MA$ where A is an invertible 3 by 3 matrix.

is a linear transformation

$$T(c_1M_1 + c_2M_2) = A^{-1}(c_1M_1 + c_2M_2)A$$
$$= c_1A^{-1}M_1A + c_2A^{-1}M_2A$$
$$= c_1T(M_1) + c_2T(M_2) \checkmark$$

Typo - should be a 3 by 3 zero matrix

kernel: $A^{-1}MA = 0$ ← zero matrix

$AA^{-1}MAA^{-1} = A0A^{-1}$ $\quad \ker(T) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

$M = 0$

range $(T) = M_{3\times3}$ since given any $B \in M_{3\times3}$,

$A^{-1}MA = B \Rightarrow AA^{-1}MAA^{-1} = ABA^{-1} \Rightarrow M = ABA^{-1}$

so $T(ABA^{-1}) = B$. This shows range $(T) = M_{3\times3}$.

So T is one-to-one, onto, and bijective.

Since $T(ABA^{-1}) = B$, $T^{-1}(B) = ABA^{-1}$.

Problem 4 (25 points)

Consider P_2, which has the following bases.

$$\mathcal{B}_1 = \{1, 1+x, 1+x+x^2\} \quad \mathcal{B}_2 = \{1-x, 2+x, 1-2x+x^2\}$$

Part (a)

Show explicitly that $\mathcal{B}_2$ is a basis for P_2. [5 points]

① span: Show that

$c_1(1-x)+c_2(2+x)+c_3(1-2x+x^2) = a+bx+cx^2$

$\begin{cases} c_1+2c_2+c_3=a \\ -c_1+c_2-2c_3=b \\ c_3=c \end{cases}$ always has a solution for all a, b, c.

$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ has $\det(A) = 3 \neq 0$ so this is true.

② linear independence: Check that $c_1(1-x)+c_2(2+x)+c_3(1-2x+x^2)=0$ only has trivial solution.

$\begin{cases} c_1+2c_2+c_3=0 \\ -c_1+c_2-2c_3=0 \\ c_3=0 \end{cases}$

This is true since $\det(A) \neq 0$.

Part (b)

Find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. [10 points]

$$[I]_{B_1\to B_2} = \begin{array}{c} \begin{array}{ccc} I(1) & I(1+x) & I(1+x+x^2) \end{array} \\ \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -2 \\ \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$1 = ((1-x)+(2+x))\frac{1}{3}$

$1+x = (2+x)-1$

$= (2+x) - \frac{1}{3}((1-x)+(2+x))$

$= -\frac{1}{3}(1-x) + \frac{2}{3}(2+x)$

$1+x+x^2 = (1-2x+x^2)+3x$

$= (1-2x+x^2)+3((1+x)-1)$

$= (1-2x+x^2)+3(-\frac{1}{3}(1-x)+\frac{2}{3}(2+x)-\frac{1}{3}(1-x) - \frac{1}{3}(2+x))$

$= -2(1-x)+(2+x)+(1-2x+x^2)$

Part (c)

Consider the linear transformation $T : P_2 \to P_2$ given by $T(f) = f'' - 2f' + f$. Find the matrix of T with respect to $\mathcal{B}_2$ and $\mathcal{B}_1$, $[T]_{\mathcal{B}_2 \to \mathcal{B}_1}$. [10 points]

$$[T]_{B_2\to B_1} = \begin{bmatrix} 4 & -1 & 13 \\ -1 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

$T(1-x) = (1-x)'' - 2(1-x)' + (1-x)$

$= 0 - 2(-1) + (1-x) = 3-x$

$= -1(1+x) + 4(1)$

$T(2+x) = (2+x)'' - 2(2+x)' + (2+x)$

$= 0 - 2(1) + (2+x) = x$

$= (1+x) + (-1)(1)$

$T(1-2x+x^2) = (1-2x+x^2)'' - 2(1-2x+x^2)' + (1-2x+x^2)$

$= 2 - 2(-2+2x) + (1-2x+x^2)$

$= 7-6x+x^2$

$= (1+x+x^2) + (6-7x)$

$= (1+x+x^2) + (-7)(1+x) + 13$

Problem 5 (25 points)

Let $M_{3\times 3}$ denote the vector space of 3 by 3 matrices. Let S be the subset of 3 by 3 matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

such that $g = 0$, $d + h = 0$, $a + e + i = 0$, $b + f = 0$, and $c = 0$ (so that the sum along any diagonal going down and to the right is zero).

Show that S is a subspace of $M_{3\times 3}$. Find $\dim(S)$, and find (with proof) a basis for S.

$$r_1\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{bmatrix} + r_2\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{bmatrix} = \begin{bmatrix} r_1a_1+r_2a_2 & r_1b_1+r_2b_2 & r_1c_1+r_2c_2 \\ r_1d_1+r_2d_2 & r_1e_1+r_2e_2 & r_1f_1+r_2f_2 \\ r_1g_1+r_2g_2 & r_1h_1+r_2h_2 & r_1i_1+r_2i_2 \end{bmatrix}$$

Check: $r_1g_1 + r_2g_2 = r_1(0) + r_2(0) = 0$ ✓

$(r_1d_1 + r_2d_2) + (r_1h_1 + r_2h_2)$

$= r_1(d_1 + h_1) + r_2(d_2 + h_2) = r_1(0) + r_2(0) = 0$ ✓

$(r_1a_1 + r_2a_2) + (r_1e_1 + r_2e_2) + (r_1i_1 + r_2i_2)$

$= r_1(a_1 + e_1 + i_1) + r_2(a_2 + e_2 + i_2) = r_1(0) + r_2(0) = 0$ ✓

$(r_1b_1 + r_2b_2) + (r_1f_1 + r_2f_2) = r_1(b_1 + f_1) + r_2(b_2 + f_2)$

$= r_1(0) + r_2(0) = 0$ ✓

$r_1c_1 + r_2c_2 = r_1(0) + r_2(0) = 0$ ✓

is a subspace

Most general form of matrix in S: $\begin{bmatrix} a & b & 0 \\ d & e & -b \\ 0 & -d & -a-e \end{bmatrix}$

$$= a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

So take the basis $B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$

Above equation shows that B spans S. So just need to check linear independence.

$$c_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 & 0 \\ c_3 & c_4 & -c_2 \\ 0 & -c_3 & -c_1-c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0$. ✓

Problem 6 (25 points)

Consider the linear transformation $T : P_3 \to P_3$ given by $T(f) = f''' - f' + f$. Find a basis for $\ker(T)$. What is the dimension of range(T)?

ker(T):

$$a + bx + cx^2 + dx^3 = f(x)$$
$$b + 2cx + 3dx^2 = f'(x)$$
$$2c + 6dx = f''(x)$$
$$6d = f'''(x)$$

$$f''' - f' + f = 6d - (b + 2cx + 3dx^2) + (a + bx + cx^2 + dx^3)$$
$$= (a - b + 6d) + (b - 2c)x + (c - 3d)x^2 + dx^3$$

$$\begin{aligned} a - b \quad + 6d &= 0 \\ b - 2c \quad &= 0 \\ c - 3d &= 0 \\ d &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & -1 & 0 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\det A = 1$ so only the trivial solution $a=b=c=d=0$.

So $\ker(T) = \{0 + 0x + 0x^2 + 0x^3 = p(x)\}$

so nullity$(T) = 0$.

Thus, since $\dim(P_3) = 4$, by Rank-Nullity Theorem, rank$(T) = 4$.

So $\boxed{\dim(\text{range}(T)) = 4}$

Problem 7 (25 points)

Part (a)

Show that there is no onto linear transformation from P_3 to $M_{3\times3}$. [10 points]

Let $T: P_3 \to M_{3\times3}$ be a linear transformation.

$\dim(\ker(T)) + \dim(\text{range}(T)) = \underset{\leftarrow \dim(P_3)}{4}$ (Rank-Nullity Theorem)

Since $\dim(\ker(T)) \geq 0$, $\dim(\text{range}(T)) \leq 4$ so

$\text{range}(T) \neq M_{3\times3}$ since $\dim(M_{3\times3}) = 9$.

So T is not onto.

Part (b)

Suppose that the linear transformation $T : P_3 \to M_{2\times2}$ is one-to-one. Is T bijective? Prove your answer. [10 points]

Yes. If T is one-to-one, $\ker(T) = \{0\}$ so $\text{nullity}(T) = 0$.

Then, $\text{nullity}(T) + \text{rank}(T) = \underset{\leftarrow 4}{\dim(P_3)}$ so $\text{rank}(T) = 4$.

So $\text{range}(T)$ is a subspace of $M_{2\times2}$ with dimension 4, so since $\dim(M_{2\times2}) = 4$, we conclude that $\text{range}(T) = M_{2\times2}$.

So T is also onto, and hence bijective.

Part (c)

Find an example of a bijective linear transformation $T : P_3 \to M_{2\times2}$. [5 points]

$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$