Math 54 Midterm 2 (Practice Exam 1)

Jeffrey Kuan

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Name: ____

SSID: ____

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **150 points**.
- Good luck!

Problem 1 (10 points)

Let $D(\mathbb{R})$ be the set of continuous functions on $\mathbb{R}$ that decay rapidly. In particular, we say that a function f is in $D(\mathbb{R})$ if and only if both of the following statements are true:

$$\lim_{x\to\infty} f(x) = 0 \text{ and } \lim_{x\to-\infty} f(x) = 0$$

This is a vector space over $\mathbb{R}$.

Part (a)

Give three examples of nonzero functions in $D(\mathbb{R})$. [3 points]

Part (b)

Show that $D(\mathbb{R})$ is closed under vector addition and scalar multiplication. What is the zero vector in $D(\mathbb{R})$? [7 points]

Problem 2 (20 points)

For each part, determine (with proof) if the set U is a subspace of the vector space V. If yes, find the dimension of U.

Part (a)

$$U = \text{the set of } a + bi \text{ where } a \text{ and } b \text{ are real numbers with } a \geq b$$
$$V = \mathbb{C}$$

Part (b)

$$U = \text{the set of polynomials in } P_4 \text{ that have a zero at } x = 2$$
$$V = P_4$$

Part (c)

$$U = \text{the set of polynomials in } P_4 \text{ such that } f(0) = 2$$
$$V = P_4$$

Problem 3 (20 points)

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations. If the linear transformation is bijective, find its inverse linear transformation.

Part (a)

The squaring map $T : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T(f) = f^2$.

Part (b)

The conjugation map $T : M_{3\times 3} \to M_{3\times 3}$ given by $T(M) = A^{-1}MA$ where A is an invertible 3 by 3 matrix.

Problem 4 (25 points)

Consider P_2, which has the following bases.

$$\mathcal{B}_1 = \{1, 1+x, 1+x+x^2\} \quad \mathcal{B}_2 = \{1-x, 2+x, 1-2x+x^2\}$$

Part (a)

Show explicitly that $\mathcal{B}_2$ is a basis for P_2. [5 points]

Part (b)

Find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. [10 points]

Part (c)

Consider the linear transformation $T : P_2 \to P_2$ given by $T(f) = f'' - 2f' + f$. Find the matrix of T with respect to $\mathcal{B}_2$ and $\mathcal{B}_1$, $[T]_{\mathcal{B}_2 \to \mathcal{B}_1}$. [10 points]

Problem 5 (25 points)

Let $M_{3\times 3}$ denote the vector space of 3 by 3 matrices. Let S be the subset of 3 by 3 matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

such that $g = 0$, $d + h = 0$, $a + e + i = 0$, $b + f = 0$, and $c = 0$ (so that the sum along any diagonal going down and to the right is zero).

Show that S is a subspace of $M_{3\times 3}$. Find $\dim(S)$, and find (with proof) a basis for S.

Problem 6 (25 points)

Consider the linear transformation $T : P_3 \to P_3$ given by $T(f) = f''' - f' + f$. Find a basis for $\ker(T)$. What is the dimension of $\text{range}(T)$?

Problem 7 (25 points)

Part (a)

Show that there is no onto linear transformation from P_3 to $M_{3\times 3}$. [10 points]

Part (b)

Suppose that the linear transformation $T : P_3 \to M_{2\times 2}$ is one-to-one. Is T bijective? Prove your answer. [10 points]

Part (c)

Find an example of a bijective linear transformation $T : P_3 \to M_{2\times 2}$. [5 points]