Math 54 Midterm 2

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Name: Answer Key

SSID: ______

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- You may assume that all matrices and values are real-valued, unless they are specifically stated to be complex-valued.
- All vector spaces are over $\mathbb{R}$.
- There are 7 questions.
- The exam has a total of **150 points**.
- Good luck!

Problem 1 (10 points)

Let V be the set of periodic sequences of real numbers with period 4, where a periodic sequence with period 4 is a sequence $a_0, a_1, a_2, a_3, a_4, a_5, \ldots$ of real numbers that repeats after every four terms. For example, the following (infinitely long) sequences are elements of V.

$$1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \ldots$$

$$-2, 0, 0, 1, -2, 0, 0, 1, -2, 0, 0, 1, \ldots$$

$$2, 0, 1, 9, 2, 0, 1, 9, 2, 0, 1, 9, \ldots$$

Part (a)

What are the operations of vector addition and scalar multiplication on V? Check that V is closed under vector addition and scalar multiplication. What is the zero vector in V? [6 points]

vector addition: add componentwise.

$$\begin{array}{r} a_1, a_2, a_3, a_4, a_1, a_2, a_3, a_4, \ldots \\ +\ b_1, b_2, b_3, b_4, b_1, b_2, b_3, b_4 \\ \hline a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4, a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4, \ldots \end{array}$$

(still has period 4)

scalar mult: multiply each component by the scalar.

$$c(a_1, a_2, a_3, a_4, a_1, a_2, a_3, a_4, \ldots)$$
$$= ca_1, ca_2, ca_3, ca_4, ca_1, ca_2, ca_3, ca_4, \ldots$$

(still has period 4)

zero vector: $0, 0, 0, 0, 0, 0, 0, 0, \ldots$

(has period 4)

Part (b)

Find, without proof, a basis for V, and $\dim(V)$. [4 points]

$$\mathcal{B} = \{1,0,0,0,1,0,0,0,\ldots,\ 0,1,0,0,0,1,0,0,\ldots,\ 0,0,1,0,0,0,1,0,\ldots,\ 0,0,0,1,0,0,0,1,\ldots\}$$

$\dim(V) = 4$.

Problem 2 (15 points)

For each part, determine (with proof) if the set U is a subspace of the vector space V.

Part (a)

$$U = \{\text{continuous functions } f \text{ in } C(\mathbb{R}) \text{ such that } f(0) \geq 0\}$$
$$V = C(\mathbb{R})$$

(Recall that $C(\mathbb{R})$ is the set of continuous functions on $\mathbb{R}$.)

not a subspace

e.g. $x^2+1 \in U$ but $(-1)(x^2+1) \notin U$

not closed under scalar multiplication

Part (b)

$$U = \{\text{polynomials } p(x) \text{ in } P_{10} \text{ such that } p(5) = 0 \text{ and } p'(2) = 0\}$$
$$V = P_{10}$$

is a subspace

If $f, g \in U$, then $c_1 f + c_2 g \in U$ too since

$(c_1 f + c_2 g)(5) = c_1 f(5) + c_2 g(5) = c_1 \cdot 0 + c_2 \cdot 0 = 0$

$(c_1 f + c_2 g)'(2) = c_1 f'(2) + c_2 g'(2) = c_1 \cdot 0 + c_2 \cdot 0 = 0$

Part (c)

$$U = \{\text{functions in } C^\infty(\mathbb{R}) \text{ such that } (f')^2 - 4f = 0\}$$
$$V = C^\infty(\mathbb{R})$$

(Hint: $f(x) = x^2$ is a function such that $(f')^2 - 4f = 0$. Recall that $C^\infty(\mathbb{R})$ is the set of smooth functions, which are functions with infinitely many derivatives.)

not a subspace

$f(x) = x^2 \in U$ since $((x^2)')^2 - 4(x^2) = (2x)^2 - 4x^2 = 0$

but $2f(x) = 2x^2 \notin U$ since $((2x^2)')^2 - 4(2x^2)$

$= (4x)^2 - 8x^2 = 8x^2 \neq 0$

not closed under scalar multiplication

Problem 3 (25 points)

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations and find a basis for the kernel and range (to save time, just for this question, you do not need to prove that the set you give is a basis). If the linear transformation is bijective, find its inverse linear transformation.

Part (a)

The map $F : \mathbb{R}^2 \to \mathbb{C}$ given by $F(a, b) = (a + 2b) + (2a - b)i$.

$$F(c_1(a_1,b_1)+c_2(a_2,b_2)) = F(c_1a_1+c_2a_2, c_1b_1+c_2b_2)$$
$$= (c_1a_1+c_2a_2+2(c_1b_1+c_2b_2)) + (2(c_1a_1+c_2a_2)-(c_1b_1+c_2b_2))i$$
$$= c_1(a_1+2b_1) + c_1(2a_1-b_1)i + c_2(a_2+2b_2) + c_2(2a_2-b_2)i$$
$$= c_1F(a_1,b_1) + c_2F(a_2,b_2)$$ is a linear transformation

ker(F): $\begin{matrix} a+2b=0 \\ 2a-b=0 \end{matrix}$ $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5 \neq 0$ so $a=b=0$ $\boxed{\ker(F)=\{(0,0)\}}$

one-to-one

basis is empty

range(F): To get $c+di$, solve $\begin{matrix} a+2b=c \\ 2a-b=d \end{matrix}$ $\begin{matrix} a=\frac{1}{5}(c+2d) \\ b=\frac{1}{5}(2c-d) \end{matrix}$ $\boxed{\text{basis } \{1, i\}}$

So $F(\frac{1}{5}(c+2d), \frac{1}{5}(2c-d)) = c+di$ $\boxed{\text{range}(F)=\mathbb{C}}$

onto

Part (b)

So T is bijective.

$$\boxed{T^{-1}(c+di) = \left(\tfrac{1}{5}(c+2d), \tfrac{1}{5}(2c-d)\right)}$$

The map $T : M_{3\times2} \to M_{3\times2}$ given by

$$T(M) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -2 \\ 2 & 1 & -1 \end{bmatrix} M$$

is a linear transformation

$$T(c_1M_1+c_2M_2) = A(c_1M_1+c_2M_2) = c_1AM_1+c_2AM_2 = c_1T(M_1)+c_2T(M_2)$$

(Hint: Find the nullspace and ~~kernel~~ columnspace of the matrix above and use it to solve this question quickly. There was a similar question on Problem Set 5.)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -2 \\ 2 & 1 & -1 \end{bmatrix}$$ Basis for Col(A): $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} R_2-R_1 \\ R_3-2R_1 \end{matrix}$$ Basis for Nul(A): $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad A\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ Av_1 & Av_2 \\ | & | \end{bmatrix}$$

Basis for ker(T): $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \right\}$

Basis for range(T): $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \right\}$

not bijective

Problem 4 (25 points)

Consider P_1, which has the bases

$$\mathcal{C}_1 = \{2 + x, 1 + 2x\} \quad \mathcal{C}_2 = \{-1 - 2x, 3 + x\}$$

Part (a)

Find the change of basis matrices from $\mathcal{C}_1$ to $\mathcal{C}_2$. [8 points]

$2+x = c_1(-1-2x) + c_2(3+x)$

$-c_1 + 3c_2 = 2 \qquad -2c_1 + 6c_2 = 4 \qquad -c_1 + 3c_2 = 2$

$-2c_1 + c_2 = 1 \qquad -2c_1 + c_2 = 1 \qquad -6c_1 + 3c_2 = 3$

$c_2 = \frac{3}{5} \qquad c_1 = -\frac{1}{5}$

$1+2x = c_1(-1-2x) + c_2(3+x)$

$-c_1 + 3c_2 = 1$

$-2c_1 + c_2 = 2 \qquad c_2 = 0 \quad c_1 = -1$

$$\boxed{[I]_{C_1 \to C_2} = \begin{bmatrix} -\frac{1}{5} & -1 \\ \frac{3}{5} & 0 \end{bmatrix}}$$

Part (b)

Let $\mathcal{B} = \{1, x, x^2, x^3, x^4, x^5\}$ be the standard ordered basis for P_5. Consider the linear transformation $\Delta\Delta : P_5 \to P_1$ (called the **Bilaplacian**) given by

$$\Delta\Delta(p(x)) = \frac{d^4}{dx^4}(p(x)) = p''''(x)$$

In particular, the Bilaplacian applied to a polynomial gives the fourth derivative of that polynomial.

Find $[\Delta\Delta]_{\mathcal{B}\to\mathcal{C}_1}$ and $[\Delta\Delta]_{\mathcal{B}\to\mathcal{C}_2}$. (Don't worry if some of your numbers are large.) [17 points]

$\Delta\Delta(1) = \Delta\Delta(x) = \Delta\Delta(x^2) = \Delta\Delta(x^3) = 0$

$\Delta\Delta(x^4) = 24 \qquad \Delta\Delta(x^5) = 120x$

$24 = 24(1) = 24\left(\frac{1}{3}(2(2+x) - (1+2x))\right)$

$= 8(2(2+x) - (1+2x))$

$= 16(2+x) - 8(1+2x)$

$120x = 40((-1)(2+x) + 2(1+2x))$

$= (-40)(2+x) + 80(1+2x)$

$24 = \frac{24}{5}((-1-2x) + 2(3+x))$

$= \frac{24}{5}(-1-2x) + \frac{48}{5}(3+x)$

$120x = -24(3(-1-2x) + (3+x))$

$= -72(-1-2x) + (-24)(3+x)$

$$[\Delta\Delta]_{B\to C_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 16 & -40 \\ 0 & 0 & 0 & 0 & -8 & 80 \end{bmatrix} \qquad [\Delta\Delta]_{B\to C_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{24}{5} & -72 \\ 0 & 0 & 0 & 0 & \frac{48}{5} & -24 \end{bmatrix}$$

Problem 5 (25 points)

Let $M_{3\times 3}$ denote the vector space of 3 by 3 matrices. Let S be the subspace of "very trace-free" matrices, defined to be the set of 3 by 3 matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

such that $a+e+i=0$, $c+e+g=0$, $b+e+h=0$, and $d+e+f=0$ (so that the sum of the entries along each of the two main diagonals and the middle column and the middle row is zero).

Find $\dim(S)$, and find (with proof) a basis for S.

General matrix in S: $\begin{bmatrix} a & b & c \\ d & e & -d-e \\ -c-e & -b-e & -a-e \end{bmatrix}$

$$= a\begin{bmatrix} 1&0&0\\0&0&0\\0&0&-1 \end{bmatrix} + b\begin{bmatrix} 0&1&0\\0&0&0\\0&-1&0 \end{bmatrix} + c\begin{bmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{bmatrix} + d\begin{bmatrix} 0&0&0\\1&0&-1\\0&0&0 \end{bmatrix} + e\begin{bmatrix} 0&0&0\\0&1&-1\\-1&-1&-1 \end{bmatrix}$$

So take $\mathcal{B} = \left\{ \begin{bmatrix} 1&0&0\\0&0&0\\0&0&-1 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&0\\0&-1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\1&0&-1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&-1\\-1&-1&-1 \end{bmatrix} \right\}$

The above equation shows that vectors in $\mathcal{B}$ span S.

So just need to check linear independence.

$$c_1\begin{bmatrix} 1&0&0\\0&0&0\\0&0&-1 \end{bmatrix} + c_2\begin{bmatrix} 0&1&0\\0&0&0\\0&-1&0 \end{bmatrix} + c_3\begin{bmatrix} 0&0&1\\0&0&0\\-1&0&0 \end{bmatrix} + c_4\begin{bmatrix} 0&0&0\\1&0&-1\\0&0&0 \end{bmatrix} + c_5\begin{bmatrix} 0&0&0\\0&1&-1\\-1&-1&-1 \end{bmatrix} = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & -c_4-c_5 \\ -c_3-c_5 & -c_2-c_5 & -c_1-c_5 \end{bmatrix} = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$$

so $c_1 = c_2 = c_3 = c_4 = c_5 = 0$. ✓

So $\mathcal{B}$ is a basis. Hence, $\boxed{\dim(S) = 5.}$

Problem 6 (25 points)

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Consider the map $F : M_{3\times3} \to M_{3\times3}$ defined by $F(B) = AB - BA$.

Part (a)

Show that F is a linear transformation. [5 points]

$$\begin{aligned} F(c_1B_1 + c_2B_2) &= A(c_1B_1 + c_2B_2) - (c_1B_1 + c_2B_2)A \\ &= c_1(AB_1 - B_1A) + c_2(AB_2 - B_2A) \\ &= c_1F(B_1) + c_2F(B_2) \end{aligned}$$

Part (b)

Find a basis for ker(F). [15 points]

$$\begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ -2x_1+x_4 & -2x_2+x_5 & -2x_3+x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} - \begin{bmatrix} 2x_1-2x_2 & x_2 & x_3 \\ 2x_4-2x_5 & x_5 & x_6 \\ 2x_7-2x_8 & x_8 & x_9 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_2 & x_2 & x_3 \\ -2x_1 - x_4 + 2x_5 & -2x_2 & -2x_3 \\ -x_7 + 2x_8 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} 2x_2 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \\ -2x_1 - x_4 + 2x_5 &= 0 \\ -2x_2 &= 0 \\ -2x_3 &= 0 \\ -x_7 + 2x_8 &= 0 \end{aligned}$$

$$\begin{aligned} x_2 &= 0 \\ x_3 &= 0 \\ 2x_1 + x_4 - 2x_5 &= 0 \\ x_7 - 2x_8 &= 0 \end{aligned}$$

x_4, x_5, x_6, x_8, x_9 free

$$x_4\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_8\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_9\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{M_3}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}}_{M_4}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_5} \right\}$$

The above computation shows $\mathcal{B}$ spans ker(F).

Check lin. ind.

$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 + c_5M_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2}c_1 + c_2 & 0 & 0 \\ c_1 & c_2 & c_3 \\ 2c_4 & c_4 & c_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_1 = c_2 = c_3 = c_4 = c_5 = 0$

so lin. ind. So $\mathcal{B}$ is a basis for ker(F).

Part (c)

Let S be the set of 3 by 3 matrices that can be written as [redacted] $AB - BA$ for some 3 by 3 matrix B, where A is the 3 by 3 matrix defined above. The set S is a subspace of $M_{3\times3}$. Calculate dim(S). [5 points]

$S = \text{range}(F)$.

From (b), nullity$(F) = 5$.

$$\text{rank}(F) + \text{nullity}(F) = \underset{\dim(M_{3\times3})}{9}$$

So rank(F) = $\boxed{\dim(S) = 4}$

Problem 7

Consider the linear transformation $T : P_4 \to \mathbb{R}^2$ given by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

Part (a)

Show that T is onto. [10 points]

$T(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad T(1-x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Since range (T) is a subspace of $\mathbb{R}^2$ containing $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, range $(T) = \mathbb{R}^2$ so T is onto.

Part (b)

Let S be the subspace of polynomials $p(x)$ in P_4 with $p(0) = 0$ and $p(1) = 0$. Find $\dim(S)$. [5 points]

(Hint: This should be quick, using the result from part (a).)

$S = \ker(T)$. $\quad \dim(S) = \text{nullity}(T)$

$= \dim(P_4) - \text{rank}(T)$

$= 5 - 2 = 3 \quad \boxed{\dim(S) = 3}$

Part (c)

Consider the polynomials $\mathcal{B} = \{x^2 - x, x^3 - x^2, x^4 - x^3\}$, which are all in S, where S is defined as in part (b). Determine if $\mathcal{B}$ is a basis for S. [10 points]

(Hint: First determine if the polynomials in $\mathcal{B}$ are linearly independent. Then, combine this with your result in (b) to get an answer quickly.)

$c_1(x^2 - x) + c_2(x^3 - x^2) + c_3(x^4 - x^3) = 0$

$-c_1 x + (c_1 - c_2)x^2 + (c_2 - c_3)x^3 + c_3 x^4 = 0$

$-c_1 = 0$
$c_1 - c_2 = 0$
$c_2 - c_3 = 0$
$c_3 = 0$

$c_1 = 0 \Rightarrow c_1 - c_2 = 0$ implies $c_2 = 0$.
$c_3 = 0$

So the polynomials in $\mathcal{B}$ are linearly independent.

So $\mathcal{B}$ consists of three linearly independent vectors in S, which has $\dim(S) = 3$, so $\mathcal{B}$ must also span S, so $\mathcal{B}$ is a basis of S.

END OF EXAM