Math 54 Midterm 1 (Practice Exam 3)

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Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 5 questions.
- Question 1 is 40 points, Question 2 is 50 points, and Questions 3, 4, and 5 are 20 points each, for a total of **150 points**.
- Good luck!

Problem 1

Give an example of each of the following, or briefly explain why the object described cannot exist. [5 pts each]

Part (a)

A 3 by 3 matrix with all entries nonzero whose reduced row echelon form is the identity matrix.

Part (b)

A nonzero matrix A such that $\det(A^4) = 0$.

Part (c)

Six distinct nonzero vectors in $\mathbb{R}^3$ whose span is not all of $\mathbb{R}^3$.

Part (d)

A nonzero 3×3 matrix A that is its own inverse, that is not the identity matrix.

Part (e)

A number c for which the following system is inconsistent.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix}$$

Part (f)

A matrix A such that A is invertible, but A^2 is not invertible.

Part (g)

Two nonzero matrices that commute with each other, none of which are diagonal.
(Hint: Consider powers.)

Part (h)

Two 3×3 invertible matrices A and B such that $A + B$ is not invertible.

Problem 2

Short answer. [10 points each]

Part (a)

Define

$$A^{-1} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Find the trace of $A^2B(A^{-1})^2$.

Part (b)

Calculate A^3 and A^{100}, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Part (c)

Calculate the determinant of

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Part (d)

Find all values λ for which the following matrix is invertible, where I is the 4×4 identity matrix.

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 2 & 0 & 0 \\ -10 & -7 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \lambda I$$

Part (e)

Write the solution of the system of equations

$$3x_1 + x_2 - x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 - x_3 = 0$$

in parametric vector form.

Problem 3

Part (a)

Calculate the inverse of the following matrix. [10 points]

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Part (b)

Use A^{-1} and matrix multiplication to find the general solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ to the linear system

$$x_1 + x_3 = b_1$$

$$2x_1 + x_3 + x_4 = b_2$$

$$x_2 - x_4 = b_3$$

$$x_2 + x_4 = b_4$$

in terms of b_1, b_2, b_3, and b_4. [10 points]

Problem 4

Part (a)

Show that the columns of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

are linearly independent and have span equal to all of $\mathbb{R}^3$ **without using the determinant**. [10 points]

Part (b)

Show that for a square 3×3 matrix, if the columns are linearly independent in $\mathbb{R}^3$, then their span is all of $\mathbb{R}^3$. *Your explanation should be relatively brief.* (Hint: Use an equivalence theorem, and the fact that the matrix is square.) [10 points]

Problem 5

Find a quadratic polynomial $f(x) = ax^2 + bx + c$ such that $f(-1) = 1$, $f(1) = 1$, and $f(2) = 0$ by converting this problem to a system of linear equations. Is such a quadratic polynomial unique? [20 points]