Math 54 Midterm 1 (Practice Exam 1)

Jeffrey Kuan

July 8, 2019

Name: ____

SSID: ____

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 5 questions.
- Question 1 is 40 points, Question 2 is 50 points, and Questions 3, 4, and 5 are 20 points each, for a total of **150 points**.
- Good luck!

Problem 1

Give an example of each of the following, or briefly explain why the object described cannot exist. [5 pts each]

Part (a)

A matrix A such that $A\mathbf{x} = \mathbf{b}$ is always inconsistent for every $\mathbf{b}$.

Part (b)

A matrix A such that $A\mathbf{x} = \mathbf{0}$ has a unique solution, but the reduced row-echelon form of A has a row of zeros.

Part (c)

3 by 3 matrices A and B that commute, where neither A nor B is the zero matrix or the identity matrix.

Part (d)

A nonzero matrix A such that $A^{2019} = A$.

Part (e)

Five vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}$ that are linearly independent in $\mathbb{R}^3$.

Part (f)

A matrix A such that $A^{-1} = \begin{bmatrix} 4 & 2 & 0 & -4 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (Hint: Calculate the determinant)

Part (g)

Invertible matrices A and B such that AB^2 is not invertible.

Part (h)

A 5×5 matrix A such that all entries of A are nonzero but $\det(A) = 0$.

Problem 2

Short answer. [10 points each]

Part (a)

Suppose you know that for some matrices A and B, we have that

$$A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

What is the inverse of BA, if it exists?

Part (b)

Find all λ such that the determinant of M is zero, where I is the 4 by 4 identity matrix.

$$M = \begin{bmatrix} 2 & 0 & 1 & 9 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 9 \end{bmatrix} - \lambda I$$

Part (c)

Write the solution to the system $A\mathbf{x} = \mathbf{b}$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Part (d)

Let A be a 3 by 3 matrix. Suppose you know that the solution to $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is unique. What is the solution to $A\mathbf{x} = \mathbf{0}$? Explain briefly.

Part (e)

Is the following matrix invertible? Note that the last row is the sum of the first three rows. Explain briefly.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 2 & 2 & 2 & 1 \end{bmatrix}$$

Problem 3

Find conditions on a, b, and c such that (a, b, c) is in $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ in $\mathbb{R}^3$, where

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ -3 \\ -6 \end{bmatrix}, \mathbf{v_4} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Give an example of a vector in $\mathbb{R}^3$ that is not in $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$. [20 points]

Problem 4

Find all vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ in $\mathbb{R}^4$ such that $\mathbf{v}^t\mathbf{w_1} = 0$, $\mathbf{v}^t\mathbf{w_2} = 0$, $\mathbf{v}^t\mathbf{w_3} = 0$, where

$$\mathbf{w_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w_2} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{w_3} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

(Hint: Solving this problem should reduce to finding the solution to a homogeneous system of equations.) [20 points]

Problem 5

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is called a **permutation matrix**.

Part (a)

Calculate $A\mathbf{v}$ where $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$. In words, in one brief sentence, describe what multiplying a vector $\mathbf{v}$ in $\mathbb{R}^4$ on the left by A does to that vector. [5 points]

Part (b)

Calculate A^{-1} using elementary row operations. Then, calculate A^2 using matrix multiplication. What do you notice? [10 points]

Part (c)

Calculate A^3, and then calculate A^{2019}. (Hint: $2019 = 3 \cdot 673$) [5 points]