Math 54 Midterm 1

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Name: Answer Key

SSID: ______________________

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 5 questions.
- Question 1 is 40 points, Question 2 is 50 points, and Questions 3, 4, and 5 are 20 points each, for a total of **150 points**.
- Good luck!

Problem 1

Give an example of each of the following, or briefly explain why the object described cannot exist. [5 pts each]

Part (a)

A 5 by 5 matrix with determinant equal to 8 that is not a diagonal matrix.

$$\begin{bmatrix} 4 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Part (b)

A 3 by 3 non-diagonal matrix A such that for every vector $\mathbf{b}$ in $\mathbb{R}^3$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Part (c)

Two invertible matrices A and B of the same size such that $ABAB$ is not invertible.

not possible, product of invertible matrices is invertible.

Part (d)

A matrix A (with real entries) such that $A^2 = \begin{bmatrix} 5 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$. (Hint: Think about what $\det(A)$ would be.)

$$\begin{vmatrix} 5 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = -1$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

det A is a real number, so not possible since $(\det(A))^2 = -1$ is impossible.

Part (e)

Five vectors whose span is all of $\mathbb{R}^2$.

$(1, 0), (0, 1), (1, 1), (2, 2), (3, 3)$

Part (f)

A homogeneous linear system of equations in four variables whose only solution is the single point $(x_1, x_2, x_3, x_4) = (2, 0, 1, 9)$.

not possible, $(0, 0, 0, 0)$ must be a solution to every homogeneous linear system

Part (g)

A 6 by 6 matrix with determinant 1 and trace 6.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Part (h)

A 3 by 2 matrix whose columns are linearly independent in $\mathbb{R}^3$ but whose rows are not linearly independent in $\mathbb{R}^2$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Problem 2

Short answer. [10 points each]

Part (a)

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Define the matrices

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -27 & 0 \\ 0 & 8 \end{bmatrix}$$

Find a matrix M such that $M^3 = ADA^{-1}$. *Write your answer as a single* 2×2 *matrix.*

$$M = A \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} A^{-1} \qquad \text{since } \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}^3 = \begin{bmatrix} -27 & 0 \\ 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & -6 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -10 \\ 0 & 2 \end{bmatrix}$$

$$\boxed{M = \begin{bmatrix} -3 & -10 \\ 0 & 2 \end{bmatrix}}$$

Part (b)

Let A, C, and L be invertible 2×2 matrices such that

$$A^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad L^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$$

Calculate $(LAC)^{-1}$.

$$(LAC)^{-1} = C^{-1}A^{-1}L^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 9 \\ -2 & 4 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -6 & 17 \\ -4 & 13 \end{bmatrix}}$$

Part (c)

Suppose that for some $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is inconsistent. Is this enough information to deduce whether $A\mathbf{x} = \mathbf{0}$ has a unique solution or infinitely many solutions? If yes, state whether $A\mathbf{x} = \mathbf{0}$ has a unique solution or infinitely many solutions. If no, explain why.

No. e.g. A has a zero row if $Ax=b$ is inconsistent, but this does not tell us about whether A has a free variable in its reduced row echelon form.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ inconsistent but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has unique solution,

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ inconsistent

but $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has infinitely many solutions.

Part (d)

Calculate $\det(A^{2019})$ where

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$\det A = 1 \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$

$= 1 \cdot 1 \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1$

$\det(A^{2019})$

$= (\det A)^{2019} = \boxed{1}$

Part (e)

Find all real numbers c (if any exist) such that $A\mathbf{x} = 0$ has a unique solution, where

$$A = \begin{bmatrix} 1 & 0 & c & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_1 - cR_3 \\ R_2 + R_3 \\ \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} R_2 - 2R_1$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \updownarrow$$

always has a free variable.

$\boxed{\text{no } c \text{ exist}}$

Problem 3

Find all x_1, x_2, x_3, and x_4 such that

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If you solve this using systems of equations, *solve the system using row reduction.*

(Interesting observation: This calculates all matrices that commute with the matrix $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$)
[20 points]

$$\begin{bmatrix} 2x_1 + x_3 & 2x_2 + x_4 \\ -x_1 & -x_2 \end{bmatrix} - \begin{bmatrix} 2x_1 - x_2 & x_1 \\ 2x_3 - x_4 & x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_2 + x_3 & -x_1 + 2x_2 + x_4 \\ -x_1 - 2x_3 + x_4 & -x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_2 + x_3 = 0 \\ -x_1 + 2x_2 + x_4 = 0 \\ -x_1 - 2x_3 + x_4 = 0 \\ -x_2 - x_3 = 0 \end{cases}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 - R_3 \\ -R_3 \\ R_1 + R_4 \end{matrix}$$

$$\to \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 - 2R_1 \to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free

$x_3 = s$, $x_4 = t$

$x_1 = -2s + t$, $x_2 = -s$

$(x_1, x_2, x_3, x_4) = (-2s + t, -s, s, t)$

$$= \boxed{s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

Problem 4

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Part (a)

Find conditions on a, b, and c such that $\mathbf{v}_5$ is in the span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. [15 points]

When does $x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

have a solution?

$$\begin{aligned} x_1 + x_2 + 2x_3 + 3x_4 &= a \\ -x_1 \quad\quad - x_3 - x_4 &= b \\ x_2 + x_3 + 2x_4 &= c \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & a \\ -1 & 0 & -1 & -1 & b \\ 0 & 1 & 1 & 2 & c \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & a+b \\ 1 & 0 & 1 & 1 & -b \\ 0 & 1 & 1 & 2 & c \end{array}\right] \begin{array}{l} R_1 + R_2 \\ -R_2 \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & a+b \\ 1 & 0 & 1 & 1 & -b \\ 0 & 0 & 0 & 0 & -a-b+c \end{array}\right] R_3 - R_1 \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & -b \\ 0 & 1 & 1 & 2 & a+b \\ 0 & 0 & 0 & 0 & -a-b+c \end{array}\right]$$

$$\boxed{a + b = c}$$

Part (b)

Find conditions on a, b, and c such that the vectors $\mathbf{v}_1, \mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}_5$ are linearly independent in $\mathbb{R}^3$. (Hint: This should be quick.) [5 points]

no such a, b, c. Five vectors cannot be linearly independent in $\mathbb{R}^3$.

Problem 5

Part (a)

Calculate the inverse of the matrix. [8 points]

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 4 & 8 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 8 & 1 & -2 & 1 \end{array}\right] R_3 - 2R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ \frac{1}{2}R_2 \\ \frac{1}{8}R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{array}\right] R_1 - R_3$$

Part (b)

- Find a quadratic polynomial $f(x) = \cancel{ax^2 + bx + c}$ $a + bx + cx^2$ such that $f(-1) = 1$, $f(1) = -1$, $f(3) = 2$, by using a linear system of equations, and then using A^{-1} above and matrix multiplication to solve the resulting linear system.

  (Note: To get full credit, you **must** solve the system using A^{-1}. You will NOT get full credit if you solve the system using row reduction.) [8 points]

- Is such a quadratic polynomial $f(x)$ satisfying $f(-1) = 1$, $f(1) = -1$, $f(2) = 2$ unique? Explain briefly in two sentences or less. (Hint: Invertible Matrix Theorem) [4 points]

(1)
$$a + b(-1) + c(-1)^2 = 1 \qquad a - b + c = 1$$
$$a + b(1) + c(1)^2 = -1 \qquad a + b + c = -1$$
$$a + b(3) + c(3)^2 = 2 \qquad a + 3b + 9c = 2$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$-\frac{5}{8} - x + \frac{5}{8}x^2$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{3}{4} & -\frac{1}{8} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{8} \\ -1 \\ \frac{5}{8} \end{bmatrix}$$

$$\boxed{f(x) = \frac{5}{8}x^2 - x - \frac{5}{8}}$$

(2) Yes, since A is invertible, so $Ax = b$ has a unique solution by the Invertible Matrix Theorem.

END OF EXAM