Math 54 Midterm 1 Review

September 30, 2019

Part 1: True/False

Directions: For each item, circle either True or False. (1 point each)

- (True/False) If the solution set to a system of equations is represented by a line in $\mathbb{R}^n$, then the system has only one free variable.
- (True/False) The dimension of P_{2019} is 2019.
- (True/False) The set of points in $\mathbb{R}^4$ such that $x_1+x_2-x_3+3x_4=0$ is a subspace of dimension 3.
- (True/False) The left nullspace of an m by n matrix A is a subspace of $\mathbb{R}^n$.
- (True/False) $(AB)^t = A^tB^t$.
- (True/False) If A and B are invertible, then $A+B$ is also invertible.
- (True/False) The m rows of the reduced row echelon form of an m by n matrix A form a basis for the row space of A.
- (True/False) Every basis for a vector space has the same number of elements.
- (True/False) There is a basis for $\mathbb{P}^2$ containing $1+x$ as one of the basis vectors.
- (True/False) The set of polynomials in P_2 whose coefficients are all integers is a subspace of P_2.
- (True/False) The set of polynomials in P_2 whose coefficient on x is equal to zero is a subspace of P_2.
- (True/False) The set of smooth solutions to the differential equation $y''=x$ is a subspace of $C^\infty(\mathbb{R})$.
- (True/False) The set of 2 by 2 matrices with determinant 0 is a subspace of $M_{2\times 2}$.
- (True/False) If $A\mathbf{x}=\mathbf{0}$ has a unique solution for a square matrix A, then the columns of A form a basis for $\mathbb{R}^n$.
- (True/False) If A and B are invertible matrices of the same size, then so is AB.

- (True/False) If the columns of an n by n matrix A are linearly independent, then the rows of A form a basis for $\mathbb{R}^n$.
- (True/False) The kernel of the evaluation map $T : P_2 \to \mathbb{R}$ defined by $T(f(x)) = f(1)$ is the set of all polynomials of degree ≤ 2 that have a root at $x = 1$.
- (True/False) The set of points in $\mathbb{R}^2$ with $x_1 \geq 0$ and $x_2 \geq 0$ is a subspace of $\mathbb{R}^2$.
- (True/False) The set of 2 by 2 matrices with $A = -A^t$ is a subspace of $M_{2\times 2}$.
- (True/False) Any linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is surjective.

Part 2: Problems

Problem 1

Suppose that

$$T(1,1) = (1,-2) \quad T(-1,1) = (2,1)$$

Find a formula for $T(x)$. Determine if T is injective, surjective, and bijective and find $T^{-1}(y)$ if it exists.

Problem 2

Find a basis for the kernel and range of the map $T : M_{2\times 2} \to \mathbb{R}$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

Problem 3

Consider the map $T : P_2 \to \mathbb{R}^3$ given by

$$T(p(x)) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$$

- Evaluate $T(x^2 - x + 1)$ and $T(x^2)$.
- Find a basis for $\ker(T)$ and $\text{range}(T)$.
- Is T bijective? If so, find a formula for T^{-1}.

Problem 4

Find conditions on a, b, and c so that (a,b,c) is in the span of $(1,1,2)$, $(-1,-2,1)$, and $(0,-1,3)$. Then, find conditions on (a,b,c) so that $(1,1,2)$, $(-1,-2,1)$, $(0,-1,3)$, and (a,b,c) are linearly independent.

Problem 5

Show that any injective linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then, show that any surjective linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible.

Problem 6

Find the change of basis matrix between $\mathcal{B}_1 = \{(1, 1), (2, 1)\}$ and $\mathcal{B}_2 = \{(-2, 3), (1, 1)\}$.