Math 54 Quiz 8 Study Guide

Conceptual

1) True : $Q^{-1} = Q^t$

2) True

3) False $\quad QQ^t = I$

$$\det(Q)\det(Q^t) = \det(I)$$
$$\det(Q^t) = \det(Q)$$
$$(\det(Q))^2 = 1$$
$$\det Q = \pm 1$$

4) True (columns are ONB, use $\cos^2\theta + \sin^2\theta = 1$)

5) True since $Tv = Av$ for a matrix A, where the columns are Te_i ← standard basis which is an ONB. So the columns of A are an ONB so A is orthogonal.

6) $\mathrm{Col}(P) = \mathrm{Range}(\mathrm{proj}_W)$ but clearly, the range of the projection onto W is W.

7) $P^1 = P$ and for $n \geq 2$, projecting with P onto W sends W to itself so further projections onto W have no effect.

Problem 1

1)(a)
$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 0 & -2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for W is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \qquad A^tA = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$$

$$(A^tA)^{-1} = \begin{bmatrix} \frac{4}{8} & -\frac{2}{8} \\ -\frac{2}{8} & \frac{3}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{8} \end{bmatrix}$$

$$A(A^tA)^{-1}A^t = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{8} \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad Pv = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}}$$

(b) $$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = w_1 \qquad v_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= (2,0,0) - \frac{2}{3}(1,1,-1) = \left(\frac{4}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

Normalize w_1 and w_2 to get an ONB.

$$\boxed{\underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}}_{u_2}}$$

$$\underbrace{(2,0,1)}_{v} = c_1 u_1 + c_2 u_2$$

$$= \frac{\langle v, u_1 \rangle}{\underbrace{\langle u_1, u_1 \rangle}_{1}} u_1 + \frac{\langle v, u_2 \rangle}{\underbrace{\langle u_2, u_2 \rangle}_{1}} u_2$$

$$= \frac{1}{\sqrt{3}} u_1 + \frac{5}{\sqrt{6}} u_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{5}{3} \\ -\frac{5}{6} \\ \frac{5}{6} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}}$$

(c)

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$QQ^t = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}} = P$$

2) $W^\perp$ has a basis $\{\underbrace{(-4,-3,-2,-1,1,2,3,4)}_{=w_1}\}$

$$\text{proj}_{W^\perp} v = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \qquad v = (2,0,1,9,2,0,1,9)$$

$$= \frac{-8+0-\cancel{2}-9+\cancel{2}+0+3+36}{16+9+4+1+1+4+9+16} w_1$$

$$= \frac{19+3}{60} w_1$$

$$= \frac{11}{30} w_1 = \left(-\frac{44}{30}, -\frac{33}{30}, -\frac{22}{30}, -\frac{11}{30}, \frac{11}{30}, \frac{22}{30}, \frac{33}{30}, \frac{44}{30}\right)$$

$$\text{proj}_W v = v - \text{proj}_{W^\perp} v$$

$$= (2,0,1,9,2,0,1,9) - \text{proj}_{W^\perp} v$$

$$= \boxed{\left(\frac{104}{30}, \frac{33}{30}, \frac{52}{30}, \frac{281}{30}, \frac{49}{30}, -\frac{22}{30}, -\frac{3}{30}, \frac{226}{30}\right)}$$

3) $\|u+v\|^2 = \langle u+v, u+v\rangle$

$$= \langle u, u+v\rangle + \langle v, u+v\rangle$$

$$= \langle u,u\rangle + \underbrace{\cancel{\langle u,v\rangle}}_{0} + \underbrace{\cancel{\langle v,u\rangle}}_{0} + \langle v,v\rangle$$

$$= \|u\|^2 + \|v\|^2$$

Same argument shows that

$$\|u+v+w\|^2 = \|u\|^2 + \|v\|^2 + \|w\|^2$$

4) $v_1 = (1,2,0) = w_1$

$v_2 = (-1,1,1)$

$v_3 = (0,0,1)$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$= (-1,1,1) - \frac{1}{5}(1,2,0)$$

$$= \left(-\frac{6}{5}, \frac{3}{5}, 1\right)$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$= v_3 - \cancel{\frac{0}{5} w_1} - \frac{1}{\frac{36}{25}+\frac{9}{25}+1} w_2 = (0,0,1) - \frac{25}{70}\left(-\frac{6}{5}, \frac{3}{5}, 1\right)$$

$$= \left(\frac{3}{7}, -\frac{3}{14}, \frac{9}{14}\right)$$

Orthoronmalze w_1, w_2, w_3 to make ON $\Rightarrow$ $\boxed{w_1 = \frac{1}{\sqrt{5}}(1,2,0) \quad w_2 = \frac{1}{\sqrt{70}}(-6,3,5) \quad w_3 = \frac{1}{\sqrt{14}}(2,-1,3)}$

$\underset{v}{(4,2,1)} = \langle v, w_1\rangle w_1 + \langle v, w_2\rangle w_2 + \langle v, w_3\rangle w_3$

$$= \boxed{\frac{8}{\sqrt{5}} w_1 + \left(-\frac{13}{\sqrt{70}}\right) w_2 + \frac{9}{\sqrt{14}} w_3}$$