Math 54 Quiz 8 Study Guide

October 29, 2019

Solve the questions below to review for this Thursday's spooky Halloween quiz!

Conceptual Questions

- True or False: Every orthogonal matrix is invertible.
- True or False: If W is a plane, then the distance from v to W is $||\text{proj}_{W^\perp} v||$.
- True or False: The determinant of an orthogonal matrix is always equal to 1.
- True or False: Every rotation matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ is an orthogonal matrix. (There are two ways to see this, abstractly or through direct computation)
- Show that if a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ sends the standard basis of $\mathbb{R}^n$ to an orthonormal basis of $\mathbb{R}^n$, then T is given by multiplication by an orthogonal matrix.
- Explain in words (conceptually, not with a proof) why for any projection matrix P for projection onto a subspace W, we have that $\text{Col}(P) = W$.
- Explain in words (conceptually, not with a proof) why $P^n = P$ for all $n \geq 1$ for any projection matrix P.

Problems

Problem 1

Find the orthogonal projection of the vector $(2, 0, 1)$ onto $W = \text{Col}(A)$ for the matrix $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & -1 & -1 \end{bmatrix}$ in two ways.

- Method 1: Find a basis for W. Use this basis to find the projection matrix P and use P to find the projection.

- Method 2: Find a basis for W and use Gram-Schmidt to get an orthonormal basis. Use the formula for coefficients in orthonormal bases to get the projection.

- Check that $P = QQ^t$, where Q is the matrix with the orthonormal basis you found from Method 2 in its columns.

Problem 2

Find the projection of $(2, 0, 1, 9, 2, 0, 1, 9)$ in $\mathbb{R}^8$ onto the plane W given by

$$-4x_1 - 3x_2 - 2x_3 - x_4 + x_5 + 2x_6 + 3x_7 + 4x_8 = 0$$

(Hint: There is an easy way to do this, by instead projecting onto $W^\perp$. Last review sheet had a formula for a basis for $W^\perp$ in the case where W is a plane.)

Problem 3

Prove the Pythagorean theorem, which states that if u and v are orthogonal,

$$||u + v||^2 = ||u||^2 + ||v||^2$$

(Hint: Bilinearity.) What if u, v, and w are an orthogonal set? Is there a nice formula for $||u + v + w||^2$?

Problem 4

Perform the Gram-Schmidt process on the vectors $(1, 2, 0)$, $(-1, 1, 1)$, and $(0, 0, 1)$. Find the coordinates of $(4, 2, 1)$ with respect to the orthonormal basis you get out from the Gram-Schmidt process.