Math 54 Quiz 6 Solutions

October 17, 2019

Question 1 (True/False)

- True. If $Av = \lambda_1 v$ and $Bv = \lambda_2 v$, then $(A+B)v = Av + Bv = (\lambda_1 + \lambda_2)v$ so v is an eigenvector of $A+B$ with eigenvalue $\lambda_1 + \lambda_2$.
- False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable over the reals or complex numbers.
- False. Again, consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- False. $2v$ is still an eigenvector of A with the same eigenvalue λ, since $A(2v) = 2(Av) = 2(\lambda v) = \lambda(2v)$.
- True. The characteristic polynomial is a real polynomial with degree 2019. So any non-real eigenvalues come in conjugate pairs. Thus, at worst, we can have 2018 complex eigenvalues, so one of the eigenvalues must be real.
- True. $1-2i$ and $-1+3i$ are also eigenvalues. There is only one eigenvalue remaining and it must be real since any non-real eigenvalues for a real-valued matrix come in conjugate pairs.
- True. Suppose A is a diagonalizable real-valued matrix whose only eigenvalue is 1. Then, $A = SDS^{-1}$ where $D = I$ since it has 1 all along the diagonal. Thus, $A = SIS^{-1} = SS^{-1} = I$.
- True. If A is not invertible, then $Av = 0$ for some nonzero vector v, or equivalently $Av = 0v$. So $\lambda = 0$ is an eigenvalue of A.
- False. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.
- True. If A is an invertible diagonalizable matrix, then $A = SDS^{-1}$ where D has nonzero diagonal entries (since 0 is not an eigenvalue of A by the invertibility of A). Thus, $A^{-1} = (SDS^{-1})^{-1} = S^{-1}D^{-1}S = (S^{-1})D^{-1}(S^{-1})^{-1}$ where D^{-1} is a diagonal matrix whose entries are the reciprocals of the entries of D.

Question 2

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

The characteristic polynomial is

$$\text{char}_A(x) = \begin{vmatrix} -x & 0 & 1 \\ 2 & 2-x & 1 \\ 0 & 0 & -2-x \end{vmatrix} = -x(-2-x)(2-x)$$

So the eigenvalues are as follows:

- $\lambda = 0$ with algebraic multiplicity 1.

$$U_0 = \text{nullspace}(A - 0I) = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

  so a basis for the eigenspace consists of the single vector $(-1, 1, 0)$. So the geometric multiplicity of $\lambda = 0$ is 1.

- $\lambda = -2$ with algebraic multiplicity 1.

$$U_{-2} = \text{nullspace}(A + 2I) = \text{nullspace} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \left\{ x_3 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

  so a basis for the eigenspace consists of the single vector $(-1/2, 0, 1)$. So the geometric multiplicity of $\lambda = -2$ is 1.

- $\lambda = 2$ with algebraic multiplicity 1.

$$U_2 = \text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} -2 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} = \left\{ x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

  so a basis for the eigenspace consists of the single vector $(0, 1, 0)$. So the geometric multiplicity of $\lambda = 2$ is 1.

Note that A is diagonalizable over the real numbers (and also over the complex numbers).

So we can write $A = SDS^{-1}$ where $S = \begin{bmatrix} -1 & -1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Thus,

$$\lambda^n A^n = S(\lambda^n D^n)S^{-1} = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & (-2\lambda)^n & 0 \\ 0 & 0 & (2\lambda)^n \end{bmatrix} S^{-1}$$

So $\lim_{n\to\infty} \lambda^n A^n$ exists when $\lim_{n\to\infty}(-2\lambda)^n$ and $\lim_{n\to\infty}(2\lambda)^n$ both exist. These limits exist when $-1/2 \leq \lambda < 1/2$ and when $-1/2 < \lambda \leq 1/2$ respectively.

So $\lim_{n\to\infty} \lambda^n A^n$ exists when $-1/2 < \lambda < 1/2$.

Question 3

Consider

$$B = \begin{bmatrix} 0 & -1 \\ 4 & k \end{bmatrix}$$

We compute

$$\text{char}_B(x) = \begin{vmatrix} -x & -1 \\ 4 & k-x \end{vmatrix} = x^2 - kx + 4$$

Use the discriminant ($D = b^2 - 4ac$) to analyze the characteristic polynomial.

- When $k^2 - 16 > 0$ (when $k > 4$ or $k < -4$), the characteristic polynomial has two real roots. So B has two distinct real eigenvalues and is diagonalizable over the real numbers (and also over the complex numbers).
- When $k^2 - 16 = 0$ (when $k = 4$ or $k = -4$), the characteristic polynomial has repeated real roots. So we need to calculate the eigenspaces to determine if this matrix is diagonalizable.
  - If $k = 4$, we have $B = \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix}$. $\lambda = 2$ has algebraic multiplicity 2 but since

    $$U_2 = \text{nullspace} \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} = \left\{ x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$$

    the geometric multiplicity is only equal to 1. So the matrix is not diagonalizable in this case over either the real or complex numbers.
  - If $k = -4$, we have $B = \begin{bmatrix} 0 & -1 \\ 4 & -4 \end{bmatrix}$. $\lambda = -2$ has algebraic multiplicity 2 but since

    $$U_{-2} = \text{nullspace} \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = \left\{ x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$$

    the geometric multiplicity is only equal to 1. So the matrix is not diagonalizable in this case over either the real or complex numbers.
- When $k^2 - 16 < 0$ (when $-4 < k < 4$), the characteristic polynomial has distinct complex roots. So in this case, B is not diagonalizable over the real numbers, but B is diagonalizable over the complex numbers.

So B is diagonalizable over the real numbers when $k > 4$ or $k < -4$, and B is diagonalizable over the complex numbers when $k \neq -4, 4$.