Math 54: Problem Set 9

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This problem set is due **August 3, 2019, 11:59 PM**. You may collaborate with other students in the class on this problem set, but your solutions must be own, and you should write up the solutions by yourself. Some of the problems below may be challenging, so start early, and feel free to come to office hours or ask questions on Piazza if you need help.

Textbook Problems

These questions are from Lay, Lay, McDonald *Linear Algebra and Its Applications*

- **Section 6.5:** 2, 4, 6, 8, 10
- **Section 7.1:** 10, 12, 14, 18, 20, 27

Additional Problems

Problem 1

Recall that if A is a symmetric matrix, then A has n eigenvalues (counted with multiplicity). Let $\lambda_{\min}$ denote the smallest eigenvalue of A and let $\lambda_{\max}$ denote the largest eigenvalue of A. Show that

$$\lambda_{\min} = \min_{v \neq 0} \frac{\langle Av, v \rangle}{\langle v, v \rangle}$$

$$\lambda_{\max} = \max_{v \neq 0} \frac{\langle Av, v \rangle}{\langle v, v \rangle}$$

If you would like, you can set $n = 3$ for concreteness. This fundamental result is called **Rayleigh's principle**. (Hint: Use the Spectral Theorem for symmetric matrices to get an orthonormal basis of eigenvectors.)

Problem 2

The **Rademacher functions** are important functions defined on the unit interval $[0, 1]$ that have important applications to probability theory. (Specifically, they model outcomes of coin flips and can be used to translate problems in probability into problems in analysis, a branch of math related to calculus.) Define the first three Rademacher functions on $[0, 1]$ as follows.

$$R_1(x) = 1 \text{ for all } 0 \le x < 1/2$$
$$R_1(x) = -1 \text{ for all } 1/2 \le x \le 1$$

$$R_2(x) = 1 \text{ for all } x \in [0, 1/4), [1/2, 3/4)$$
$$R_2(x) = -1 \text{ for all } x \in [1/4, 1/2), [3/4, 1]$$

$$R_3(x) = 1 \text{ for all } x \in [0, 1/8), [1/4, 3/8), [1/2, 5/8), [3/4, 7/8)$$
$$R_3(x) = -1 \text{ for all } x \in [1/8, 1/4), [3/8, 1/2), [5/8, 3/4), [7/8, 1]$$

Graph $R_1(x), R_2(x), R_3(x)$. Check that $\{R_1(x), R_2(x), R_3(x)\}$ is an orthonormal set in $L^2([0,1])$. (Hint: To calculate the integrals, use the definition of the integral in terms of areas.)

Problem 3

Show that if A is a positive definite matrix, then all diagonal entries of A must be positive. Then, show that if A is a positive semidefinite matrix, then all diagonal entries of A must be nonnegative.

(Hint: Consider $\langle Av, v\rangle$ for a particular choice of v.)

Problem 4

Determine which of the following matrices are positive definite.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix} \quad C = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(Hint: Use Sylvester's criterion.)

Problem 5

Recall that Sylvester's criterion can be used to show that a matrix is positive definite. Show that the analogue of Sylvester's criterion does not hold for positive semidefinite matrices. Specifically, find an n by n symmetric matrix A such that each j by j upper left submatrix has nonnegative determinant for $1 \le j \le n$, but A is not positive semidefinite.

Problem 6

(Taken from Fall 1985 UC Berkeley Ph.D. Mathematics Preliminary Examination)

Let k be real, n an integer ≥ 2, and let $A = (a_{ij})$ be the $n \times n$ matrix such that: all diagonal entries $a_{ii} = k$, all entries $a_{ii\pm 1}$ immediately above or below the diagonal equal 1; all other entries equal 0. For example, if $n = 5$,

$$A = \begin{bmatrix} k & 1 & 0 & 0 & 0 \\ 1 & k & 1 & 0 & 0 \\ 0 & 1 & k & 1 & 0 \\ 0 & 0 & 1 & k & 1 \\ 0 & 0 & 0 & 1 & k \end{bmatrix}$$

Let $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and largest eigenvalues of A, respectively. Show that $\lambda_{\min} \leq k - 1$ and $\lambda_{\max} \geq k + 1$.

(Hint: Use Rayleigh's principle, where you choose a particular v for each case.)