Math 54: Problem Set 8

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This problem set is due **July 31, 2019, 11:59 PM**. You may collaborate with other students in the class on this problem set, but your solutions must be own, and you should write up the solutions by yourself. Some of the problems below may be challenging, so start early, and feel free to come to office hours or ask questions on Piazza if you need help.

Textbook Problems

These questions are from Lay, Lay, McDonald *Linear Algebra and Its Applications*

- **Section 6.1:** 13, 16, 18, 24, 29, 31
- **Section 6.2:** 5, 6, 10, 12, 14, 16, 26
- **Section 6.3:** 6, 10, 12, 13, 15
- **Section 6.4:** 7, 8, 10, 12 (For 10 and 12, find an **orthonormal basis** instead of an orthogonal basis for the column space)

Additional Problems

Problem 1

In class, we talked about isometric linear transformations in $\mathbb{R}^n$. We will consider the general case of an abstract linear operator.

Let V be an abstract inner product space (not necessarily $\mathbb{R}^n$). Define an isometric linear operator $T : V \to V$ to be an invertible (bijective) linear operator that preserves inner products, in the sense that

$$\langle v, w \rangle = \langle Tv, Tw \rangle$$

- Prove the polarization identity:

$$\langle v, w \rangle = \frac{1}{2} \left(\langle v + w, v + w \rangle - \langle v, v \rangle - \langle w, w \rangle \right)$$

- Use the polarization identity to prove that if T is a bijective linear operator from V to V, then T is an isometric linear transformation if and only if T preserves norms (which means that $||Tv|| = ||v||$ for all $v \in V$).
- Show that the inverse $T^{-1} : V \to V$ of an isometric linear operator is also an isometric linear operator.
- Show that the identity transformation $I : V \to V$ is an isometric linear operator.
- Show that if $T_1 : V \to V$ and $T_2 : V \to V$ are both isometric linear operators, then the composition $T_2 \circ T_1 : V \to V$ defined by $(T_2 \circ T_1)(v) = T_2(T_1(v))$ is also an isometric linear operator.

Remark: The last three parts above show that the set of isometric linear operators from V to V form an algebraic structure called a **group**.

Problem 2

Let A be an n by n matrix. Show that if the columns of A form an orthonormal basis for $\mathbb{R}^n$, then the rows of A form an orthonormal basis for $\mathbb{R}^n$ too. (Hint: What kind of matrix is A? What does this imply about A^{-1}? Use $AA^{-1} = A^{-1}A = I$.)

Problem 3

Without using any linear algebra and by only using the Pythagorean theorem, find the length of the diagonal of a rectangular box with dimensions 2, 3, and 5. Use this example to explain why the formula $||v|| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ for the norm of a vector in $\mathbb{R}^3$ can be interpreted as the length of the vector.

Problem 4

$\mathcal{B} = \{v_1 = (1, 0, 0), v_2 = (1, 1, 1), v_3 = (1, -1, 1)\}$ is a basis for $\mathbb{R}^3$. Let $\ell : \mathbb{R}^3 \to \mathbb{R}$ be a linear functional such that $\ell(v_1) = 2$, $\ell(v_2) = -1$, $\ell(v_3) = 1$. By the Riesz representation theorem, $\ell(v) = \langle v, w\rangle$ for some fixed vector w. Find w.

Problem 5

Suppose that T is a linear operator on $\mathbb{R}^n$. Let $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ be an orthonormal basis for $\mathbb{R}^n$. We say that T is a **symmetric linear transformation** if

$$\langle Tv, w\rangle = \langle v, Tw\rangle \tag{1}$$

for all v and w in V.

- Show that if T is a symmetric linear transformation, then $[T]_{\mathcal{B}\to\mathcal{B}}$ is a symmetric matrix.
- Show that if $[T]_{\mathcal{B}\to\mathcal{B}}$ is a symmetric matrix, then T is a symmetric linear transformation (so it satisfies (1)). (Hint: Show (1) is true when v and w are basis vectors in the orthonormal basis $\mathcal{B}$ for $\mathbb{R}^n$, and use bilinearity to get the result for general v and w in $\mathbb{R}^n$.)

Problem 6

Recall that given an (abstract) linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, we can define the abstract transpose linear transformation $T^t : \mathbb{R}^n \to \mathbb{R}^m$ by

$$\langle Tv, w \rangle = \langle v, T^t w \rangle \tag{2}$$

for all $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$. Let T be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ -2 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Denote $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ -2 & 0 & 1 & 2 \end{bmatrix}$ and let $\mathcal{B}$ and $\mathcal{C}$ denote the standard bases with respect to $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

- Calculate the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$, $[T]_{\mathcal{B}\to\mathcal{C}}$.
- Calculate $T^t(1, 0, 0)$, $T^t(0, 1, 0)$, and $T^t(0, 0, 1)$ using the definition (2) above. Use this to find the matrix $[T^t]_{\mathcal{C}\to\mathcal{B}}$. What do you notice about the matrix of T compared to the matrix of T^t?

Problem 7

Perform the Gram-Schmidt process to find an orthonormal basis for $W = \text{Span}\{1, x, x^2\}$ in $L^2([0, 1])$.