Math 54: Problem Set 7

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This problem set is due **July 25, 2019, 11:59 PM**. You may collaborate with other students in the class on this problem set, but your solutions must be own, and you should write up the solutions by yourself. Some of the problems below may be challenging, so start early, and feel free to come to office hours or ask questions on Piazza if you need help.

Textbook Problems

These questions are from Lay, Lay, McDonald *Linear Algebra and Its Applications*

- **Section 5.1:** 13, 16, 18, 19, 25, 26
- **Section 5.2:** 14, 16, 18, 20
- **Section 5.3:** 4, 6, 15, 20, 27, 32

Additional Problems

Problem 1

Recall the Fibonacci sequence, defined by $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. We will take $a_0 = 1$ and $a_1 = 1$ for this problem.

- Recall that there is a matrix M such that

$$\begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = M \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix}$$

  Diagonalize M as ADA^{-1}. (Incidentally, the **golden ratio** should end up being one of the eigenvalues. There will be square roots involved here.)

- Use your diagonalization to find a closed form formula for the 1001th Fibonacci number, a_{1000} (since a_0 is the first Fibonacci number). If you are curious, you can plug your answer into a computer, though the number might be too big to avoid roundoff error.

Problem 2

Let V be the vector space of infinite sequences of real numbers $a_0, a_1, a_2, ...$ Let $T : V \to V$ be the shift operator, defined by

$$T(a_0, a_1, a_2, ...) = (a_1, a_2, a_3, ...)$$

Find all eigenvalues of T and eigenvectors of T.

Problem 3

First, let $T : V \to V$ be a linear operator on a finite-dimensional vector space V.

- Show that if λ is an eigenvalue of T, then $T - \lambda I$ is not invertible.
- Show that if $T - \lambda I$ is not invertible (so it is either not onto, or not one-to-one), then λ is an eigenvalue of T. (Hint: Consider first the case where $T - \lambda I$ is not one-to-one, in which case the result is immediate. Then, consider the case where $T - \lambda I$ is not onto. Use Additional Problem 1 in Problem Set 6.)

Now, let $T : V \to V$ be a linear operator on an infinite dimensional vector space V.

- Show that if λ is an eigenvalue of T, then $T - \lambda I$ is not invertible. (So the first statement is still true in this case.)
- Show that if $T - \lambda I$ is not invertible, it is not necessarily true that λ is an eigenvalue of T. Why is this so, considering that the result is true for finite dimensional vector spaces? (Hint: Consider a variant of the linear transformation in Problem 2, given by $T(a_0, a_1, a_2, ...) = (0, a_0, a_1, a_2, ...)$ and consider $\lambda = 0$.)

Problem 4

Consider the linear transformation $T : P_3 \to P_3$ given by $T(f) = \frac{d}{dx}(xf(x))$.

- Find the matrix of T with respect to any basis of your choice.
- Use your previous result to find the trace, determinant, characteristic polynomial, eigenvalues, and eigenvectors of the abstract linear transformation T.