# Math 54: Problem Set 6

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This problem set is due **July 20, 2019, 11:59 PM**. You may collaborate with other students in the class on this problem set, but your solutions must be own, and you should write up the solutions by yourself. Some of the problems below may be challenging, so start early, and feel free to come to office hours or ask questions on Piazza if you need help.

## Textbook Problems

These questions are from Lay, Lay, McDonald *Linear Algebra and Its Applications*

- **Section 4.4:** 4, 8, 14, 19
- **Section 4.7:** 6, 8, 14

## Additional Problems

### Problem 1

Let $V$ be a finite-dimensional vector space. A **linear operator** is any linear transformation from a vector space $V$ to itself. Let $T : V \to V$ be a linear operator, where $\dim(V)$ is finite.

- Show that if $T$ is one-to-one, then $T$ is onto (and hence bijective).
- Show that if $T$ is onto, then $T$ is one-to-one (and hence bijective).
- For each part above, give an example (two examples total) to show that the results above do not necessarily hold if $T$ is just a linear transformation between two different vector spaces (and hence is not a linear operator).

(Hint: Using Rank-Nullity Theorem will make the first two parts very short.)

### Problem 2

Here is more practice on kernel, range, rank, and nullity.

- Show that if $T : V \to W$ is a linear transformation, then $\text{range}(T)$ is a subspace of $W$.

- Consider $M_{n\times n}$. Let $S$ be the subspace of $n\times n$ matrices whose elements in the first row sum to zero, and whose elements in the second column sum to zero. Find $\dim(S)$. (Hint: Define a linear transformation $T : M_{n\times n} \to \mathbb{R}^2$ such that $S = \ker(T)$ and use Rank-Nullity Theorem).
- Consider $P_{40}$. Let $S$ be the subspace of polynomials in $P_{40}$ whose coefficients on the even powers all sum to zero. Find $\dim(S)$. (Hint: Similar idea to the previous part.)

## Problem 3

A vector space is called **infinite dimensional** if no finite set of vectors spans the whole space. Let $\mathbb{R}^\infty$ be the set of infinite sequences of real numbers $a_0, a_1, a_2, ...$ that are eventually zero (so that there exists a positive integer $M$ such that the $M$th term and beyond are all zeros).

- Check that $\mathbb{R}^\infty$ is a vector space. (For the sake of brevity, you only need to verify that it is closed under addition, closed under scalar multiplication, has a zero vector, and has a notion of additive inverse, and not the other properties.)
- Show that $\mathbb{R}^\infty$ is infinite dimensional. (Hint: Argue by contradiction. Suppose that finitely many sequences $S_1, S_2, ..., S_k$ span all of $\mathbb{R}^\infty$. We will show that this is impossible. There is some large enough positive integer $N$ for which the $N$th term and beyond for all of these finitely many sequences is all zeros. But then the span of these sequences has $N$th term and beyond all equal to 0, so the span cannot be all of $\mathbb{R}^\infty$, which gives a contradiction).

## Problem 4

Recall that if $V$ is a vector space, then $L(V, \mathbb{R})$ is the set of all linear transformations from $V$ to $\mathbb{R}$, which is called the set of **linear functionals on** $V$. Describe all elements of $L(\mathbb{R}^n, \mathbb{R})$.

(Remark: You may notice that these linear transformations are all given by dot products with a fixed vector in $\mathbb{R}^n$. This is a simple example of a mathematical result called the **Riesz representation theorem**. More on this when we talk about inner product spaces.)

## Problem 5

Consider the sets $M_{3\times3}$ and $\text{Skew}_{3\times3}$, where $\text{Skew}_{3\times3}$ is the subspace of skew symmetric matrices (matrices such that $A^t = -A$). Let $\mathcal{B}$ denote the standard ordered basis for $M_{3\times3}$ and consider the sets

$$\mathcal{C}_1 = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

$$\mathcal{C}_2 = \left\{ \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

- You already saw in Problem Set 4 that $\mathcal{C}_1$ is an ordered basis for $\text{Skew}_{3\times3}$. Prove that $\mathcal{C}_2$ is an ordered basis for $\text{Skew}_{3\times3}$ also (can be done easily using a system of equations and the determinant) and find the change of basis matrix from $\mathcal{C}_1$ to $\mathcal{C}_2$ and the change of basis matrix from $\mathcal{C}_2$ to $\mathcal{C}_1$.

- For any matrix $A$ in $M_{3\times3}$, show that $A - A^t$ is in $\text{Skew}_{3\times3}$. Then, prove that the function $T : M_{3\times3} \to \text{Skew}_{3\times3}$ given by $T(A) = A - A^t$ is a linear transformation. Then, find $[T]_{\mathcal{B}\to\mathcal{C}_1}$ and $[T]_{\mathcal{B}\to\mathcal{C}_2}$ where $\mathcal{B}$ is the standard basis for $M_{3\times3}$. (Hint: To do this very quickly, it will help to find the inverse of the matrix $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and apply this to find the solution to a system of equations. You may use a calculator to do this, and for all matrix multiplications.)

- Consider the transpose linear transformation $t : \text{Skew}_{3\times3} \to \text{Skew}_{3\times3}$. Find $[t]_{\mathcal{C}_1\to\mathcal{C}_1}$, $[t]_{\mathcal{C}_1\to\mathcal{C}_2}$, $[t]_{\mathcal{C}_2\to\mathcal{C}_1}$ and $[t]_{\mathcal{C}_2\to\mathcal{C}_2}$. Check that $I = [t]^2_{\mathcal{C}_1\to\mathcal{C}_1} = [t]^2_{\mathcal{C}_2} = [t]_{\mathcal{C}_2\to\mathcal{C}_1}[t]_{\mathcal{C}_1\to\mathcal{C}_2} = [t]_{\mathcal{C}_1\to\mathcal{C}_2}[t]_{\mathcal{C}_2\to\mathcal{C}_1}$. Why does this make sense, considering the abstract definition (not matrix definition) of the transpose? (Hint: What is $t$ composed with $t$?)

- Next, note that it is not true that $I = [t]^2_{\mathcal{C}_1\to\mathcal{C}_2}$ or $I = [t]^2_{\mathcal{C}_2\to\mathcal{C}_1}$. Why does this not contradict your explanation in the previous part? (Hint: Carefully keep track of which basis the coordinates are with respect to when you do $t$ composed with $t$.)