Math 54: Problem Set 5

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This problem set is due **July 17, 2019, 11:59 PM**. You may collaborate with other students in the class on this problem set, but your solutions must be own, and you should write up the solutions by yourself. Some of the problems below may be challenging, so start early, and feel free to come to office hours or ask questions on Piazza if you need help.

Textbook Problems

These questions are from Lay, Lay, McDonald *Linear Algebra and Its Applications*

- **Section 4.3:** 8, 11, 16, 23
- **Section 4.5:** 9, 12, 16
- **Section 4.6:** 2, 6, 13, 16

Additional Problems

Problem 1

Recall that P_n is the set of polynomials with real coefficients with degree $\leq n$. Show that $\{1, 1+x+2x^2, -2+2x^2, 1+2x-x^2+x^3\}$ is a basis for P_3.

(Remark: Recall in class that we showed that $\{1, x, x^2, x^3\}$ is a basis for P_3, but this question shows that one vector space has many choices for a basis. But either way, the number of elements in any basis - the dimension - is always the same).

Problem 2

Let $C(\mathbb{R})$ the vector space of all continuous functions on $\mathbb{R}$. Show that the functions 1, $\sin(x)$, $\cos(x)$, $\sin(2x)$, and $\cos(2x)$ are linearly independent in $C(\mathbb{R})$.

(Hint: Formulate this is a linear system in 5 equations and 5 variables, then use the determinant.)

(Remark: In particular, the collection of all $\sin(kx)$ and $\cos(kx)$ for $k = 0, 1, 2, ...$ are all linearly independent in $C(\mathbb{R})$, so $C(\mathbb{R})$ is an infinite dimensional vector space.)

Problem 3

Let $M_{3\times3}$ denote the vector space of 3 by 3 matrices with real entries. What is the dimension of $M_{3\times3}$?

Determine if the following sets are subspaces of $M_{3\times3}$. Concisely prove your answer. If yes, find the dimension of the subspace and an appropriate basis. Prove that the set you found is a basis. (**Note:** This question is not as long as it looks, since a good portion of the following sets are not subspaces.)

- $\mathbb{Z}_{3\times3}$ = the set of matrices with integer entries.
- $Diag$ = the set of diagonal matrices.
- Sym = the set of symmetric matrices ($A^t = A$).
- $Skew$ = the set of skew-symmetric matrices ($A^t = -A$).
- N = the set of matrices with all entries nonnegative.
- TrF = the set of trace-free matrices ($\mathrm{Tr}(M) = 0$).
- Inv = the set of invertible matrices.
- NS = the set of matrices with nonzero nullspace.
- C = the set of matrices that commute with $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 4

For each linear transformation, find the kernel and range. Determine if the linear transformation is one-to-one, onto, and bijective. If the linear transformation is bijective, calculate the inverse linear transformation.

- The cyclic permutation linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_2, x_3)$.
- The inclusion linear transformation $i : \mathbb{R} \to \mathbb{R}^2$ given by $i(x) = (x, 0)$.
- The projection linear transformation $P : \mathbb{R}^3 \to \mathbb{R}^2$ given by $P(x_1, x_2, x_3) = (x_1, x_2)$.
- The conjugation linear transformation $T : M_{3\times3} \to M_{3\times3}$, given by $T(M) = AMA^{-1}$, where A is a fixed invertible matrix.
- Let V be the vector space of sequences of real numbers $a_0, a_1, a_2, ...$ (How would you add two sequences together and multiply a sequence by a real number? The sum of the sequences $a_0, a_1, a_2, ...$ and $b_0, b_1, b_2, ...$ is $a_0 + b_0, a_1 + b_1, a_2 + b_2, ...$)

Define the Fibonacci linear transformation $F : \mathbb{R}^2 \to V$ where $F(a_0, a_1)$ is the Fibonacci sequence that starts with a_0, a_1 (where every term is the sum of the two before it). For example,

$$F(1,1) = 1, 1, 2, 3, 5, 8, 13, ...$$

$$F(2,0) = 2, 0, 2, 2, 4, 6, 10, ...$$

$$F(4,-3) = 4, -3, 1, -2, -1, -3, -4, -7, ...$$

Answer the questions above for the linear transformation F.

Problem 5

Define the linear transformation $T : M_{3\times 4} \to M_{5\times 4}$ by

$$T(M) = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} M$$

- What is a basis for the column space of $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and its dimension? What is a basis for the nullspace of A and its dimension? What is $\text{rank}(A) + \text{nullity}(A)$? Does this agree with the rank-nullity theorem?
- Find a basis for $\ker(T)$ and $\text{range}(T)$. (Note that this is different from the previous part.) Justify your result, but you do not need to prove it in detail. (Hint: Even though this is different from your previous part, if you think of what the matrix multiplication means, this part can be solved easily using the previous part.)
- What is $\dim \ker(T) + \dim \text{range}(T)$? Does this agree with the rank-nullity theorem?

Problem 6

Recall from Problem Set 2 the Vandermonde matrix, which for given real numbers a_1, a_2, ..., a_n is defined to be the n by n matrix

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

Recall the fact that the Vandermonde matrix has nonzero determinant whenever the numbers a_1, a_2, ..., a_n are distinct.

Let x_1, x_2, ..., x_n be distinct real numbers. (If the abstract math notation here is confusing, you are free to set $n = 3$ throughout the problem.)

- Show that there exists a unique polynomial f_1 of degree $\leq n-1$ such that $f_1(x_1) = 1$ and $f_1(x_k) = 0$ for $k = 2, ..., n$. (Hint: Invertible Matrix Theorem)

- More generally, show that there exists polynomials $f_1, f_2, ..., f_n$ each of degree $\leq n-1$ such that

  $$f_i(x_i) = 1, \quad f_i(x_k) = 0 \text{ if } i \neq k$$

  Show also that these polynomials are unique. These polynomials are called **Lagrange polynomials** or **interpolating polynomials**.

- Note that $f_1, f_2, ..., f_n$ are elements of P_{n-1}. Show that they form a basis for P_{n-1}. This shows in a different way (see Problem 1) that $\dim(P_{n-1}) = n$.

  To do this (to show that the polynomials span P_n), you may use the fact that through any n points with distinct x values, there is exactly one polynomial of degree $\leq n-1$ (this is easy to show using the Invertible Matrix Theorem, but I know this problem is already long enough).

  (Hint: Use the property in part (b). For example, to show linear independence, we need to show that

  $$c_1 f_1 + c_2 f_2 + ... + c_n f_n = 0$$

  implies $c_1 = c_2 = ... = c_n = 0$. To do this, evaluate the above equation at x_1 to get $c_1 = 0$. Do the same for $x_2, ..., x_n$. Do something similar to show that the span of $f_1, f_2, ..., f_n$ is P_{n-1}, using the fact mentioned above.)

Problem 7

Let us be more concrete about Problem 6. Let $x_1 = -1$, $x_2 = 0$, $x_3 = 1$.

- Calculate the Lagrange polynomials f_1, f_2, and f_3. What degree are these polynomials?

- Suppose we know that f is some polynomial such that $f(-1) = 2$, $f(0) = 3$, and $f(1) = -1$. Write f as a linear combination of f_1, f_2, and f_3. (This should be short). Use this linear combination to find what f is. Using the fact that $\{f_1, f_2, f_3\}$ is a basis of P_2, explain how you know that there is only one polynomial f such that $f(-1) = 2$, $f(0) = 3$, and $f(1) = -1$.

The second part of this question shows why these polynomials are called interpolating polynomials. This is because **interpolation** is the process of finding a curve that passes through a certain number of given points. In the second part of the question, we found a quadratic polynomial that passed through the points (-1, 2), (0, 3), and (1, -1).