Math 54: Problem Set 4

Jeffrey Kuan

July 3, 2019

This problem set is due **July 11, 2019, 11:59 PM**. You may collaborate with other students in the class on this problem set, but your solutions must be own, and you should write up the solutions by yourself. Some of the problems below may be challenging, so start early, and feel free to come to office hours or ask questions on Piazza if you need help.

Textbook Problems

These questions are from Lay, Lay, McDonald *Linear Algebra and Its Applications*

- **Section 4.1:** 2, 7, 20, 21, 31, 33
- **Section 4.2:** 3, 12, 14, 22, 33

Additional Problems

Problem 1

For any nonnegative integer n, let P_n denote the set of polynomials with degree $\leq n$ with real coefficients (where we include the zero polynomial). For example, P_0 is the set of constant polynomials (e.g. $y = 1$, $y = 2$, $y = -4$). As we discussed in class P_n is a vector space.

- Let U_n be the set of polynomials with degree $\leq n$ with constant term equal to zero. Is U_n a subspace of P_n? Prove your answer. (For example, $2x^3 - 3x^2 + x$ is in U_3)
- If V_n is the set of polynomials with degree $\leq n$ that have at least one coefficient equal to zero, is V_n a subspace of P_n? Prove your answer. (For example, $4x^3 + 3x - 2$ is in V_3 since the coefficient on x^2 is 0.)
- If E_n is the set of even polynomials with degree $\leq n$ (even meaning that $f(-x) = f(x)$), is E_n is a subspace of P_n? Prove your answer. (For example, $x^2 \in E_3$.)
- If W_n is the set of polynomials with degree $\leq n$ whose coefficients all add to zero, is W_n a subspace of P_n? Prove your answer. (For example, $x^3 - 3x^2 + 2$ is in W_3 since $1 + (-3) + 0 + 2 = 0$.)

- Let $n \geq 2$. We showed in class that the derivative map $\frac{d}{dx} : P_n \to P_{n-1}$ given by $\frac{d}{dx}(f) = f'$ is a linear transformation. Calculate $\ker(D)$ and $\text{range}(D)$. You should be able to express your answers succinctly using the definitions in the previous parts of this question.

- Let $n \geq 1$. We showed in class that the evaluation at zero map $E_0 : P_n \to \mathbb{R}$ given by $E_0(f) = f(0)$ is a linear transformation. Calculate $\ker(E_0)$ and $\text{range}(E_0)$. You should be able to express your answers succinctly using the definitions in the previous parts of this question.

Problem 2

Check if the following maps are linear transformations between vector spaces or not. If yes, find the kernel and range.

- The determinant map $\det : M_{3\times 3} \to \mathbb{R}$.

- The transpose map $t : M_{3\times 2} \to M_{2\times 3}$ that takes a 3×2 matrix A to the matrix A^t.

- The map from $F : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

- The cyclic permutation map $P : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$P(x_1, x_2, x_3, x_4) = (x_4, x_1, x_2, x_3)$$

- The multiplication map $M : \mathbb{R}^2 \to \mathbb{R}$ given by $M(x_1, x_2) = x_1 x_2$.

- The division map $D : C(\mathbb{R}) \to C(\mathbb{R})$, given by $D(g) = \frac{g(x)}{x^2+1}$, where $C(\mathbb{R})$ is the vector space of continuous functions.

Problem 3

Let V be a vector space over $\mathbb{R}$, and let $L(V, \mathbb{R})$ denote the set of linear transformations from V to $\mathbb{R}$. $L(V, \mathbb{R})$ is called the set of **(real) linear functionals on** V.

- Show that $L(V, \mathbb{R})$ is a vector space, by giving it operations of vector addition and scalar multiplication, and checking the necessary properties.

- Describe $L(\mathbb{R}, \mathbb{R})$. (The description should be an easy one, though getting there might be a bit conceptually difficult.)

Problem 4

Let $C^1(\mathbb{R})$ denote the vector space of differentiable functions with continuous derivative defined on $\mathbb{R}$ (why is this a vector space?), and let $C(\mathbb{R})$ denote as usual the vector space of continuous functions on $\mathbb{R}$. Show that the differential operator

$$\left(\frac{d}{dx} - 4\right) : C^1(\mathbb{R}) \to C(\mathbb{R})$$

is a linear transformation, where $\left(\frac{d}{dx} - 4\right) f(x) = f'(x) - 4f(x)$.

For 0.5 bonus points, find the kernel and range of the linear transformation $\left(\frac{d}{dx} - 4\right)$. (For the kernel, you need separation of variables, and for the range, you need to solve a linear first order differential equation by using an integrating factor.)