Math 54: Problem Set 3

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This problem set is due **July 6, 2019, 11:59 PM**. You may collaborate with other students in the class on this problem set, but your solutions must be own, and you should write up the solutions by yourself. Some of the problems below may be challenging, so start early, and feel free to come to office hours or ask questions on Piazza if you need help.

Textbook Problems

These questions are from Lay, Lay, McDonald *Linear Algebra and Its Applications*

- **Section 3.1:** 12, 13, 19, 20, 21, 37, 40
- **Section 3.2:** 9, 20, 28, 39, 40
- **Section 1.4:** 22
- **Section 1.5:** 37
- **Section 1.7:** 12
- **Section 2.1:** 17, 22
- **Section 2.2:** 18

Additional Problems

Problem 1

Label the following statements as true or false. Justify your answers (if true, explain why the statement is true; if false, give a counterexample).

- A homogeneous system of equations always has at least one solution.
- If A and B are n by n matrices, then $\det(A+B)$ equals $\det(A)+\det(B)$.
- If A and B are n by n matrices, then $\text{trace}(A+B)$ equals $\text{trace}(A)+\text{trace}(B)$.
- If P is a matrix with determinant -1, then $\det(P^2AP^{10})$ equals $\det(A)$.

- If $A^2 = I$ where A is a matrix with real entries, then $\det(A) = 1$.
- If $A^3 = I$ where A is a matrix with real entries, then $\det(A) = 1$.

Problem 2

Suppose that for each $j = 1, 2, ..., n$, we have that

$$\sum_{i=1}^{n} a_{ij} = 0$$

for real numbers a_{ij}. (This means that the sum of the entries in each column of the coefficient matrix is zero.) Does the homogeneous system of equations

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = 0$$

in the variables x_1, x_2, ..., x_n have a unique solution? Justify your answer. (Don't use specific examples to justify your answer; your justification should be in terms of the general system above)

Hint: If the notation is confusing to you, an example of such a system is

$$3x_1 + 2x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$-4x_1 - 3x_2 - 2x_3 = 0$$

since $3 + 1 + (-4) = 2 + 1 + (-3) = 1 + 1 + (-2) = 0$.

Problem 3

- Show that $\det(A) = \det(A^t)$ for a square matrix A.
- Then, show that for a square matrix A, if the columns of A are linearly independent, then the rows of A are also linearly independent. (**Hint:** I suggest that you use the fact that $\det(A) = \det(A^t)$ to do this, but you can use a different method if you want.)

Problem 4

A **nilpotent matrix** is a square matrix A such that $A^n = 0$ for some positive integer n. What is the determinant of A, where A is any nilpotent matrix? Prove your answer. Find an example of a nonzero 2 by 2 nilpotent matrix.

Problem 5

Show by example that if A and B are square matrices of the same size, then the difference of squares formula

$$(A+B)(A-B) = A^2 - B^2$$

is not necessarily true. Find a condition on A and B such that $(A+B)(A-B) = A^2 - B^2$, and justify your answer. (Hint: FOIL. The condition is a simple one.)

Problem 6

Let A be an $m \times m$ square matrix, let B be an $n \times n$ square matrix, and let C be an $m \times n$ matrix, where m and n are not necessarily equal. Suppose that A and B (which, recall, are square) are invertible. Show that the **block matrix**

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

is invertible, where 0 here is an $n \times m$ zero matrix.

Hint: A block matrix is a matrix whose components are made up of other matrices. As an example of the result that you need to show above (where $m = 3$ and $n = 2$), the 3 by 3 matrix A and the 2 by 2 matrix B below are both invertible.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$$

If we let C be the 3 by 2 matrix

$$C = \begin{bmatrix} -3 & 1 \\ -2 & 4 \\ 3 & 2 \end{bmatrix}$$

then the claim is that the block matrix

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & -3 & 1 \\ 0 & 1 & -1 & -2 & 4 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

is invertible, where here, the 0 in the block matrix is a 2 by 3 zero matrix. As a hint, think about these questions: What is the reduced row echelon form of A, and the reduced row echelon form of B? Using this, what would be the reduced row echelon form of M?