Math 54: Problem Set 2

Jeffrey Kuan

June 28, 2019

This problem set is due **July 3, 2019, 11:59 PM**. You may collaborate with other students in the class on this problem set, but your solutions must be own, and you should write up the solutions by yourself. Some of the problems below may be challenging, so start early, and feel free to come to office hours or ask questions on Piazza if you need help.

Textbook Problems

These questions are from Lay, Lay, McDonald *Linear Algebra and Its Applications*

- **Section 1.7:** 2, 8, 10, 34, 36, 38
- **Section 2.1:** 1, 8, 9, 10, 27
- **Section 2.2:** 3, 6, 14, 22, 32
- **Section 2.3:** 6, 13, 19, 28

Additional Problems

Problem 1

For two n by n matrices A and B, define the **commutator** $[A, B]$ by

$$[A, B] = AB - BA$$

- Define the matrices

$$X = \begin{pmatrix} 2 & 1 & 3 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad Y = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 2 & -2 \end{pmatrix}$$

  Calculate $[X, Y]$.

- Show that $[A, A] = 0$. Then, show that if A and B commute, then $[A, B] = 0$.
- Prove that $[A, B] = -[B, A]$ for all n by n matrices A and B. This property is called **antisymmetry**.

- Suppose now that A and B are both diagonal matrices. Calculate $[A, B]$. Justify your answer.

- The commutator also satisfies the following fundamental property, called the **Jacobi identity**, which states that

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

for any n by n matrices A, B, and C. Check that this identity holds for the following 2 by 2 matrices. (If you want, you can also prove this identity algebraically fairly easily, though it is not required.)

$$A = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

Problem 2

The Fibonacci sequence generated by two real numbers a_0 and a_1 is defined as follows. Given any two numbers a_0 and a_1, we define

$$a_n = a_{n-1} + a_{n-2}, \text{ for } n \geq 2$$

For example, if $a_0 = 1$ and $a_1 = 1$, the resulting Fibonacci sequence is

$$a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8, a_6 = 13, a_7 = 21, ...$$

- Find a matrix F such that for $n \geq 2$,

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = F \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix}$$

- Calculate F^{32}. (Hint: You only need five matrix multiplications.) Use this to find a formula for a_{32} and a_{33} in terms of only a_0 and a_1. Note: You may use a calculator for the arithmetic, but do not let the calculator do the actual matrix multiplication out for you.

Problem 3

The following n by n matrix is known as the **Vandermonde matrix**, where $a_1, a_2, ..., a_n$ are given real numbers.

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

Here, n can be any positive integer. Note that this is a square n by n matrix. For example, if $a_1 = 1, a_2 = 3, a_3 = 5, a_4 = 6$, then the associated Vandermonde matrix is the 4 by 4 matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \end{pmatrix}$$

- Show that the Vandermonde matrix is not invertible if any two of the numbers $a_1, a_2, ..., a_n$ is the same.
- It is a well-known fact in linear algebra that the Vandermonde matrix is invertible if and only if the numbers $a_1, a_2, ..., a_n$ are distinct. Use this fact to prove the following important fact.

  If a polynomial of degree less than or equal to $n - 1$ has n distinct roots,

  it must be the zero polynomial.

  (Hint: Convert this problem into a linear algebra problem involving a linear system with n variables and n equations)

Problem 4

Consider the two matrices

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Calculate A^{-1} and D^{-1}. Then, explicitly calculate $(A^{-1}DA)^{2019}$.