Math 54: Problem Set 11

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August 11, 2019

This problem set is due **August 13, 2019, 11:59 PM**. You may collaborate with other students in the class on this problem set, but your solutions must be own, and you should write up the solutions by yourself. Some of the problems below may be challenging, so start early, and feel free to come to office hours or ask questions on Piazza if you need help.

Textbook Problems

These questions are from Nagle, Saff, and Snider.

- **Section 9.5:** 12*. 15
- **Section 9.6:** 6*, 13*
- Non-textbook problem*: Solve the system

$$x_1'(t) = x_1(t) - 3x_2(t)$$

$$x_2'(t) = 3x_1(t) - 5x_2(t)$$

- **Section 10.3:** 11, 12, 13, 19, 20, 21

* Use a computer to plot the phase portraits for these systems. Include pictures of these phase portraits in your final submission. Explain briefly why the phase portrait makes sense when compared to eigenvalues and eigenvectors you calculated. Is $(0,0)$ is a stable or unstable equilibrium?

Additional Problems

Problem 1

Solve the following third order differential equation by converting it to a first order system of differential equations.

$$x''' - x'' - x' + x = 0$$

Find a particular solution such that $x(0) = 1$, $x'(0) = -1$, and $x''(0) = 0$. Answer the same questions for

$$x''' + x'' + x' + x' = 0$$

Problem 2

Show that the Fourier series map defined on $C^\infty(S^1)$ interchanges differentiation with multiplication in the following sense:

$$\widehat{f'}(k) = ik\hat{f}(k)$$

for all integers k. (Hint: Integrate by parts. Use periodicity to note that the boundary terms in the integration by parts disappear.) $C^\infty(S^1)$ means $C^\infty([-\pi, \pi])$, where $f(\pi) = f(-\pi)$. (S^1 is a mathematical notation for a circle).

Use this fact to show that for any $f \in C^\infty(S^1)$, the function $\hat{f}(k)$ defined on $\mathbb{Z}$ is **rapidly decreasing** in the following sense: for any positive integer N, there is a constant c_N (depending on N) such that $|\hat{f}(k)| \leq \frac{C}{k^n}$ so that $|\hat{f}(k)|$ decays faster than k to *any* negative power.

(Hint: First iterate the above result to show that

$$k^N \widehat{f}(k) = (-i)^N \widehat{f^{(N)}}(k)$$

where $f^{(N)}$ is the Nth derivative of f. Then, show that $|k^N \widehat{f}(k)|$ is always less than or equal to some constant C_N. To do this, note that

$$|k^N \widehat{f}(k)| = |\widehat{f^{(N)}}(k)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f^{(N)}(\theta) e^{-ik\theta} d\theta \right|$$

Use the Triangle Inequality for integrals, $\left|\int f(x)dx\right| \leq \int |f(x)|dx$ and the Extreme Value Theorem from calculus.)

Problem 3

Adapted from Elias M. Stein and Rami Shakarchi's book, *Fourier Analysis: An Introduction (Princeton Lectures in Analysis I)*

In class, we used the Fourier series for $f(\theta) = \theta$ on $[-\pi, \pi]$ and Plancherel's formula to show that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. (This is an important calculation, so if you don't remember it, please review it!). We outline yet another proof of this fact in the problem below.

Show that for the function $f(\theta) = |\theta|$ on $[-\pi, \pi]$, extended to be a continuous periodic function with period 2π on $\mathbb{R}$,

$$\hat{f}(0) = \frac{\pi}{2} \quad \text{and } \hat{f}(n) = \frac{\cos(n\pi) - 1}{\pi n^2} \text{ for } n \neq 0$$

(Hint: Integrate by parts. This gets a little computationally messy, so be careful, especially with your signs! Don't forget to use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ somewhere along the way.)

Using the calculation above, evaluate the Fourier series identity

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(n) e^{-in\theta}$$

at $\theta = 0$ (how do you know the series converges to f? - mention this in your solution) to deduce that

$$\sum_{k\geq 1, k \text{ odd}} \frac{1}{k^2} = \frac{\pi^2}{8}$$

Finally, use this fact to show the fundamental result that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

As a remark, this shows that $\zeta(2) = \frac{\pi^2}{6}$, where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

(defined initially for $s > 1$, where this series converges) is the **Riemann zeta function**.

Problem 4

Taken from UC Berkeley Ph.D Preliminary Exam, Fall 1997

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be distinct real numbers. Show that the n exponential functions $e^{\alpha_1 t}, e^{\alpha_2 t}, ..., e^{\alpha_n t}$ are linearly independent over the real numbers.

(Hint: Differentiate n times and use the Vandermonde matrix. You may set $n = 3$ if you would like.)

Problem 5

Taken from UC Berkeley Ph.D Preliminary Exam, Spring 1994

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a diagonalizable linear transformation. Prove that there is an orthonormal basis for $\mathbb{R}^n$ with respect to which T has an upper-triangular matrix.

(Hint: You may set $n = 2$ if you would like. There is a basis of n eigenvectors that spans $\mathbb{R}^n$. Apply Gram-Schmidt to get a new orthonormal basis for $\mathbb{R}^n$, which now does not necessarily consist of eigenvectors. Is T written with respect to this new basis orthogonal?)

Problem 6

Taken from UC Berkeley Ph.D Preliminary Exam, Spring 1990

Let n be a positive integer, and let $A = (a_{ij})_{i,j=1}^{n}$ be the $n \times n$ matrix with $a_{ii} = 2$, $a_{ii\pm 1} = -1$, and $a_{ij} = 0$ otherwise; that is

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Prove that every eigenvalue of A is a positive real number.

(Hint: First show that A always has real eigenvalues, n of them in fact. Then, show that every eigenvalue is positive by using Rayleigh's principle with $v = (1, 1, 0, 0, ..., 0)$.)

Problem 7 (Bonus: 0.25 points, Multivariable Calculus required)

Define the convolution operation on $[0, 2\pi]$ by

$$(f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y)g(x-y)dy$$

The Fourier series map interchanges convolution of functions and products of functions.

- Let f and g be integrable on $[0, 2\pi]$. Show that
$$\widehat{(f * g)}(k) = \widehat{f}(k)\widehat{g}(k)$$
- Show that
$$\widehat{fg}(k) = \sum_{n \in \mathbb{Z}} \hat{f}(n)\hat{g}(k-n)$$
  Note that the right hand side is a discrete version of convolution.

Problem 8 (Bonus: 0.25 points, Multivariable Calculus required)

Consider the wave equation

$$\partial_t^2 u - \Delta u = 0$$

with initial conditions $u(0, x) = f(x)$ and $\partial_t u(0, x) = g(x)$, where f and g are smooth functions. Let us consider this equation in one time and one spatial dimension, on $\mathbb{R}^+ \times \mathbb{R}$, so that the equation is really just

$$\partial_t^2 u - \partial_x^2 u = 0$$

- Show that $(\partial_t + \partial_x)(\partial_t - \partial_x)u = \partial_t^2 u - \partial_x^2 u$.
- Solve the wave equation in this case, by first solving
$$(\partial_t + \partial_x)v = 0$$
  and then solving $(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0$.

  (Hint: Consider a directional derivative. Integrate along slanted lines. A partial differential equation that gives a condition on a directional derivative is called a **transport equation**.)