Math 1A: Examples of $\epsilon - \delta$ Proofs

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1 Example 1

Question: Prove using an $\epsilon - \delta$ proof that

$$\lim_{x \to 2} (x+1) = 3$$

Proof.

- Consider any $\epsilon > 0$.
- We want to find a $\delta > 0$ so that if $0 < |x-2| < \delta$, then $|(x+1) - 3| < \epsilon$.
- Simplifying, we want to find a $\delta > 0$ so that if $0 < |x-2| < \delta$, then $|x-2| < \epsilon$.
- But we see that if we just let $\delta = \epsilon$, then this would exactly work, since if $0 < |x-2| < \delta$, then $|x-2| < \delta$ and $|x-2| < \epsilon$, as $\delta = \epsilon$. So we have shown that if $0 < |x-2| < \delta$ for $\delta = \epsilon$, then $|x-2| < \epsilon$.
- So we are done.

□

2 Example 2

Question: Prove using an $\epsilon - \delta$ proof that

$$\lim_{x \to -1} (3x+1) = -2$$

Proof.

- Consider any $\epsilon > 0$.
- We want to find a $\delta > 0$ so that if $0 < |x+1| < \delta$, then $|(3x+1) - (-2)| < \epsilon$.
- Simplifying, we want to find a $\delta > 0$ so that if $0 < |x+1| < \delta$, then $|3x+3| < \epsilon$.
- Simplifying even more, we want to find a $\delta > 0$ so that if $0 < |x+1| < \delta$, then $3|x+1| < \epsilon$.
- But if we just let $\delta = \frac{\epsilon}{3}$, then this is true. Because if $0 < |x+1| < \delta$, since $\delta = \frac{\epsilon}{3}$, then $|x+1| < \frac{\epsilon}{3}$. Multiplying both sides by 3, we have that $3|x+1| < \epsilon$, as desired. So we have shown that for $\delta = \frac{\epsilon}{3}$, if $|x+1| < \delta$, then $3|x+1| < \epsilon$.
- So we are done.

□

3 Example 3

Question: Consider the function

$$f(x) = 1 \text{ if } x = 0$$

$$f(x) = 0 \text{ if } x \neq 0$$

Use an $\epsilon - \delta$ proof to show that $\lim_{x\to 0} f(x) = 0$.

Proof.

- Consider any $\epsilon > 0$.
- We want to find a $\delta > 0$ so that if $0 < |x - 0| < \delta$, then $|f(x) - 0| < \epsilon$.
- Simplifying, we want to find a $\delta > 0$ so that if $0 < |x| < \delta$, then $|f(x)| < \epsilon$.
- We note that any δ here will work, because if $0 < |x| < \delta$, then $x \neq 0$, so that $f(x)$ will always be 0.
- To illustrate this, we can let $\delta = 1$. We show that this works. If $0 < |x| < 1$, then certainly, $x \neq 0$ and thus, $f(x) = 0$, so that $|f(x)| = 0 < \epsilon$. So we have shown that if $0 < |x| < 1$ (where $\delta = 1$), then $|f(x)| = 0 < \epsilon$.
- So we are done.

□

4 Example 4

Question: Consider the function

$$f(x) = x^5 + 2$$

Use an $\epsilon - \delta$ proof to show that f is continuous at the origin.

Proof. To show that f is continuous at the origin, we must show that

$$\lim_{x\to 0} f(x) = f(0)$$

Since $f(0) = 2$, all we need to do is show that $\lim_{x\to 0} x^5 + 2 = 2$. We can do this using an $\epsilon - \delta$ proof.

- Consider any $\epsilon > 0$.
- We want to find a $\delta > 0$ so that if $0 < |x - 0| < \delta$, then $|(x^5 + 2) - 2| < \epsilon$.
- Simplifying, we want to find a $\delta > 0$ so that if $0 < |x| < \delta$, then $|x^5| < \epsilon$.
- We note that if we let $\delta = \sqrt[5]{\epsilon}$, then this will work. The next step is to check that this choice of δ works.
- If $0 < |x| < \delta$, then since $\delta = \sqrt[5]{\epsilon}$, we have that $0 < |x| < \sqrt[5]{\epsilon}$. So $|x| < \sqrt[5]{\epsilon}$. This implies that $|x|^5 < \epsilon$. Since $|x|^5 = |x^5|$, we have that $|x^5| < \epsilon$ whenever $0 < |x| < \delta$. So we have shown that if $0 < |x| < \delta$ where $\delta = \sqrt[5]{\epsilon}$, then $|x^5| < \epsilon$.
- So we are done.

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