Math 54: Linear Algebra and Differential Equations

Jeffrey Kuan

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Today, we will look at a new way of solving systems of equations, $A\mathbf{x} = \mathbf{b}$. Let's go back to basic algebra. If we wanted to solve the following equation

$$2x = 4$$

what would we do? We would multiply both sides by the multiplicative inverse of 2, which is 1/2, and get

$$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 4 \Longrightarrow x = 2$$

So if we had some sort of inverse for matrices, then we might be able to solve

$$A\mathbf{x} = \mathbf{b}$$

by multiplying by the "multiplicative inverse" of A, A^{-1}. But to do this, we would need to properly make sense of how to multiply matrices together.

Basic Matrix Operations

So let's learn how to do basic matrix operations. Remember that we say a matrix is of size $\mathbf{m} \times \mathbf{n}$ if it has m rows and n columns. We say that the $(\mathbf{i}, \mathbf{j})$ **entry** is the entry in the ith row and the jth column. If $m = n$ (number of rows equals the number of columns), we say the matrix is a **square matrix**. A square matrix that has all non-diagonal entries equal to zero is called a **diagonal matrix**.

Adding matrices and multiplying them by scalars is done in the way you would expect, componentwise. But you can only add matrices of the **same size**. So for example,

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix} + 2\begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & -2 & 1 \end{bmatrix}$$

The entry in the $(2, 1)$ position in the resulting matrix is 6.

There is also another matrix operation called the **transpose**, which when applied to an $m \times n$ matrix, gives an $n \times m$ matrix where the (i, j) entry in the original matrix goes to the (j, i) entry in the new matrix. So for example,

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}^t = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Note that the second matrix in the previous example is its own transpose. We say that a matrix A is **symmetric** if $A^t = A$.

Now, let's consider matrix multiplication. Matrix multiplication follows a weird pattern, but one that makes sense once you know about compositions of linear transformations.

We can only matrix multiply two matrices A and B if A has size $m \times n$ and B has $n \times r$. The resulting matrix AB is then $m \times r$. So in particular, **the first matrix you multiply must have the same number of columns as rows of the second matrix.**

So if A is a 1×4 matrix and B is a 4×1 matrix, then AB is 1×1 and BA is 4×4. In particular, note that **matrix multiplication is not commutative**; in particular, it is **NOT** always true that $AB = BA$ (and most of the times it is not).

How do you multiply matrices? Let's start with the simplest example of a matrix that has one row and a matrix that has one column. To do this multiplication, we use the following pattern.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} = [1 \cdot 2 + 2 \cdot (-1) + (-1) \cdot (-3) + 0 \cdot 1] = [3]$$

where $[3]$ is the resulting 1×1 matrix that we get out.

To generalize this to matrices of more complicated sizes, we use this "add the products along the row of the first matrix and column of the second matrix" pattern in the following way.

- Determine what the size of the resulting matrix should be.
- To get the (i, j) entry in the multiplied matrix, consider row i of the first matrix and column j of the second matrix.
- Use the "add the products along the row of the first matrix and column of the second matrix" pattern along the ith row of the first matrix and the jth column of the second matrix.

For example,

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + (-1) \cdot 1 & 2 \cdot 1 + (-1) \cdot 2 \\ 1 \cdot 3 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 3 & 1 \end{bmatrix}$$

Note that with matrix multiplication, the notation $A\mathbf{x} = \mathbf{b}$ for systems makes sense now, as $A\mathbf{x}$ can be thought of as matrix multiplication.

Problem 1

Matrix multiplication is not commutative, even if AB and BA have the same size. Check that $AB \neq BA$ for the following matrices

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

However, matrix multiplication is **associative**. In particular, $ABC = (AB)C = A(BC)$. So we can do matrix multiplications however we want, *as long as we do not change the order of the matrices we are multiplying!*

Problem 2

When we multiply real numbers, when we multiply anything by 1, the number does not change. For matrix multiplication, what is the identity? It turns out that a multiplicative identity exists for matrix multiplication. The identity for square matrix multiplication is a **diagonal matrix** that has 1 all along the diagonal. For example, the 3×3 identity matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check that for the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

we have that $AI = A$ and $IA = A$.

Problem 3

$AB = 0$ where 0 is the zero matrix does not imply that either $A = 0$ or $B = 0$. To see this, compute

$$\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

Problem 4

How does taking a transpose affect matrix multiplication? Let A and B be defined as

$$A = \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}$$

- Calculate $(AB)^t$.
- One might think that the formula is $(AB)^t = A^t B^t$ **but this is wrong!** What is $A^t B^t$ for the matrices above (if it even makes sense).
- The correct formula is
$$(AB)^t = B^t A^t$$
Calculate $B^t A^t$ to see this explicitly.

Problem 5

Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \\ 1 & -1 & -1 \end{bmatrix}$$

Check that $AB = I$ and $BA = I$. B is called the **inverse matrix** of A, and we write $B = A^{-1}$.

Matrix Inverse

Some matrices have inverses. In particular, for **square matrices**, the notion of matrix inverse makes sense. Not all square matrices have inverses but some of them do.

Let's say that a square matrix A of size $n \times n$ has an inverse A^{-1}. By definition, this means that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

where I is the $n \times n$ identity matrix. Then, if we wanted to solve the system

$$A\mathbf{x} = \mathbf{b}$$

we could solve it by doing

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \Longrightarrow \mathbf{x} = A^{-1}\mathbf{b}$$

This motivates something called the **invertible matrix theorem**.

Invertible Matrix Theorem: These following two statements are equivalent for a **square** matrix A.

- The square matrix A has an inverse A^{-1}.
- The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.
- The homogeneous system $A\mathbf{x} = 0$ only has the trivial solution

Warning: The equivalence between statement 2 and 3 is only true for square matrices (Why?).

Problem 6

Using the result of Problem 5, which shows that for the square matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

we have that the matrix inverse is

$$A^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \\ 1 & -1 & -1 \end{bmatrix}$$

solve the system of equations

$$x_1 + x_2 = 3$$

$$2x_2 - x_3 = -1$$

$$x_1 - x_2 = 2$$

Also, verify the equivalence in the invertible matrix theorem by checking that the homogeneous problem

$$x_1 + x_2 = 0$$

$$2x_2 - x_3 = 0$$

$$x_1 - x_2 = 0$$

only has the trivial solution.

So we see that the matrix inverse is definitely useful. But how do you calculate it? The first important result is the following. **A square matrix has an inverse if and only if its reduced row echelon form has a pivot in every row** (and thus necessarily a pivot in every column too, since it is square).

How do you find the matrix inverse of a matrix A? Here are the steps.

- Make an augmented matrix with A on one side and the identity matrix of the same size on the other side $[A \mid I]$.
- Row reduce to make the side with A into reduced row echelon form.
- After this row reduction, if the left side of the line does not have a pivot in each row (is the identity matrix), the original matrix A does not have an inverse.
- If the left side of the line does have a pivot in each row (is the identity matrix), the inverse A^{-1} is what is on the right side of the line.

Problem 7

Use the algorithm above to show that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$$

does not have a matrix inverse. Then, verify the invertible matrix theorem by showing that the homogeneous system

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 - x_2 = 0$$

$$x_1 - 2x_2 - x_3 = 0$$

does not have a unique solution.