Math 54: Linear Algebra and Differential Equations

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Homogeneous and Inhomogeneous Systems

Remember that we can express solution sets in terms of using **parametric vector form**. For example, let's consider the following two systems.

$$x + 3y + z = 2$$

$$2x + 3y - 4z = 1$$

$$x + 3y + z = 0$$

$$2x + 3y - 4z = 0$$

and let us express the solution sets to each in parametric vector form. Note the first system is **non-homogeneous** and has the solution set

$$(-1 + 5s, 1 - 2s, s) = (-1, 1, 0) + s(5, -2, 1)$$

and the second system is the corresponding **homogeneous system** for the first system and has the solution set

$$(5s, -2s, s) = s(5, -2, 1)$$

We see an interesting pattern here. The solution to the non-homogeneous problem is

$$(-1 + 5s, 1 - 2s, s) = (-1, 1, 0) + s(5, -2, 1)$$

and note that the homogeneous solution $s(5, -2, 1)$ is added onto $(-1, 1, 0)$ to get the non-homogeneous solution. Here, $(-1, 1, 0)$ is a **particular solution** for the inhomogeneous problem, since if we plug it into the system

$$x + 3y + z = 2$$

$$2x + 3y - 4z = 1$$

we see that it is actually a solution. This is a general phenomenon. **The solution to an inhomogeneous system is any particular solution to the inhomogeneous system plus the solution to the homogeneous system.** This is true because you can show that the difference between any two particular solutions to a non-homogeneous system is a solution of the homogeneous system.

Equivalence Theorem for Span and $\mathbf{Ax} = \mathbf{b}$

Yesterday, we learned the concept of **span** and the **coefficient matrix** associated with a system. We will see that these concepts are actually connected to each other. The following equivalence theorem shows that span, *existence* of solutions to systems, and matrices are all connected.

Equivalence Theorem for Span: The following are equivalent:

- (1) $A\mathbf{x} = \mathbf{b}$ has a solution for EVERY column vector b.
- (2) If you row reduced A (just A, not the whole augmented matrix) to reduced row-echelon form, there is a pivot in every row.
- (3) Every column vector $\mathbf{b}$ is in the span of the columns of A (so that the span of the columns of A is $\mathbb{R}^m$, where m is the number of equations).

The fact that these are equivalent means that if any one of these statements is true, they are all true and if any one of these statements is false, they are all false. Let's illustrate this through some simple examples.

Example 1 (all three statements are true):

Suppose that we consider the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

We claim that for this matrix A, all three statements above are true.

- Let us first check that (2) is true. If we row reduce just A, we get

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  So the reduced row-echelon form of A (and JUST A, not the whole augmented matrix) has a pivot in every row. This is (2), so (2) is true.

- We can also see (1) is true. Why is this so? Well, if $\mathbf{b} = (b_1, b_2)$ is anything, then we get after row reducing to reduced-row echelon form (for the AUGMENTED matrix now)

$$\begin{bmatrix} 1 & 0 & | & * \\ 0 & 1 & | & * \end{bmatrix}$$

  where * denote some numbers (specifically, if you do the calculation out), from the augmented matrix, you get

$$\begin{bmatrix} 1 & 2 & | & b_1 \\ -1 & 1 & | & b_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & | & b_1 \\ 0 & 3 & | & b_1 + b_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & | & b_1 \\ 0 & 1 & | & \frac{1}{3}b_1 + \frac{1}{3}b_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & | & \frac{1}{3}b_1 - \frac{2}{3}b_2 \\ 0 & 1 & | & \frac{1}{3}b_1 + \frac{1}{3}b_2 \end{bmatrix}$$

So we can see that no matter what $\mathbf{b} = (b_1, b_2)$ is, there is always a solution to $A\mathbf{x} = \mathbf{b}$ (so (1) is true).

- Then finally, (3) is true since the system of equations $A\mathbf{x} = \mathbf{b}$ is the same as

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So the fact that (1) is true ($A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$) shows that $\mathbf{b}$ is a linear combination of (1, -1) and (2, 1) for every $b = (b_1, b_2)$ in $\mathbb{R}^2$. So (3) is true. So we see that for this example, all of (1), (2), and (3) are all true.

Example 2 (all three statements are false):

Now consider the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- (2) is false. If we row-reduce JUST the matrix A (not the augmented matrix), we get the following reduced row-echelon form

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

and note that there is no pivot in the second row, so (2) is false.

- We can then see that (1) is false. So it is not true that $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$. This is not to say that $A\mathbf{x} = \mathbf{b}$ always has no solutions. It just means that not every $\mathbf{b}$ will give a consistent system for $A\mathbf{x} = \mathbf{b}$. In particular, we can choose some $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ does not have a solution. To see this, note that since the row-reduced echelon form of A is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

the row-reduced echelon form of any augmented matrix for $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 1 & 2 & | & * \\ 0 & 0 & | & * \end{bmatrix} \text{ or more explicitly } \begin{bmatrix} 1 & 2 & | & b_1 \\ 0 & 0 & | & b_2 - 2b_1 \end{bmatrix}$$

where * can be any numbers. Now, if we choose $\mathbf{b} = (b_1, b_2)$ so that the bottom * ends up being a zero (in particular, if $b_2 = 2b_1$), we would indeed have a consistent system (with infinitely many solutions). But the reason (2) is false here is because we could also choose b so that the bottom * ends up being nonzero, so that our system has no solutions. So for this particular choice of $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ would have no solutions. So it is not true that $A\mathbf{x} = \mathbf{b}$ has solutions for every $\mathbf{b}$.

- Then, (3) is false. Because for the specific $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solutions, we have that the vector form of the system

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

has no solutions for this particular $\mathbf{b} = (b_1, b_2)$, so that this particular b is not in the span of the columns of A, so in particular the span of the columns of A is not all of $\mathbb{R}^2$.

These examples show the equivalence of (1), (2), and (3). Either all three conditions are true, or all three conditions are false. There are no other possibilities.

In particular, why is this theorem useful? We can use it to show that the span of vectors is or is not all of $\mathbb{R}^m$, or we can use it to show that $A\mathbf{x} = \mathbf{b}$ does have a solution for every $\mathbf{b}$ or not. **In particular, we can reduce both of these questions to row-reducing a matrix A to reduced row-echelon form.**

Linear Independence

We say that the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_k}$ are **linearly independent** if

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_k\mathbf{v_k} = 0$$

implies that $c_1 = c_2 = ... = c_k = 0$.

If $c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_k\mathbf{v_k} = 0$ for c_i not all equal to zero, then the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_k}$ are **linearly dependent**.

Example:

- $(1, 0)$ and $(0, 1)$ are linearly independent in $\mathbb{R}^2$.
- $(1, 1, -1)$, $(2, 1, 0)$, and $(0, -1, -3)$ are linearly independent in ^{3}.
- $(2, 4)$ and $(-1, -2)$ are linearly dependent in $\mathbb{R}^2$.
- $(2, 4)$, $(-1, -2)$, and $(1, 3)$ are linearly dependent in $\mathbb{R}^2$.
- $(1, 1, -1)$, $(2, 1, 0)$ and $(0, -1, 2)$ are linearly dependent in $\mathbb{R}^3$.
- If any of the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}$ is zero, then the vectors are linearly dependent.

Just as we had an equivalence theorem for span, we can reinterpret linear independence also in terms of systems. In particular, we have the following equivalence theorem for linear independence, which relates linear independence of vectors to uniqueness of solutions to systems of equations to matrix operations.

Equivalence Theorem for Linear Independence: The following conditions are equivalent (either all true or all false):

- (1) $A\mathbf{x} = \mathbf{0}$ has a unique solution.
- (2) If you row reduced A (just A, not the whole augmented matrix) to reduced row-echelon form, there are no free columns.
- (3) The columns of A are linearly independent.

This reduces the problem of whether a homogeneous system is unique, and the problem of whether given vectors are linearly independent to row-reduction of a matrix to reduced row-echelon form.

Problem 1

Show that if $k > n$, then any k vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}$ in $\mathbb{R}^n$ are linearly dependent in $\mathbb{R}^n$.