Math 54: Linear Algebra and Differential Equations

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Vectors in $\mathbb{R}^n$

The set of **linear combinations** of vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}$ is the set of vectors of the form

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_k\mathbf{v_k}$$

for real numbers c_i. This set is called the **span** of $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}$.

Example: Here are some examples of linear combinations of vectors.

- The set of linear combinations of a single vector is a line.
- The set of linear combinations of $(1, 0)$ and $(0, 1)$ is all vectors in $\mathbb{R}^2$.
- The set of linear combinations of $(2, 1)$ and $(-1, 3)$ is all vectors in $\mathbb{R}^2$.
- The zero vector is always a linear combination of any vectors.

Problem 1

Determine if the vectors $(0, 0, 0)$, $(1, 1, 0)$, and $(1, -6, 5)$ are linear combinations of the two vectors $\mathbf{v_1} = (1, 2, -1)$ and $\mathbf{v_2} = (2, 0, 1)$.

Another way that vectors are useful is that we can use them to parametrize objects such as lines. The set of vectors $\mathbf{v} + c\mathbf{w}$ for real numbers c traces out the line that passes through the point represented by $\mathbf{v}$ that goes in the direction of $\mathbf{w}$.

Example: The set of vectors of the form

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + 2s \\ 3 + s \end{bmatrix}$$

is a line that passes through $(-1, 3)$ that points in the direction of the vector $(2, 1)$.

More on Linear Systems

We can write a linear system

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

As shorthand, we can write in shorthand as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

so that the system can be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{b}$ is a linear combination of the columns of A interpreted as vectors.

The following result is important. The following conditions are equivalent:

- (1) $A\mathbf{x} = \mathbf{b}$ has a solution for EVERY column vector b.
- (2) If you row reduced A (just A, not the whole augmented matrix) to reduced row-echelon form, there is a pivot in every row.
- (3) Every column vector $\mathbf{b}$ is in the span of the columns of A (so that the span of the columns of A is $\mathbb{R}^m$, where m is the number of equations).

Finally, we can express solution sets in terms of vectors. For example, we can do this for the example yesterday.

Example:

$$x + 2y + z = 1$$

$$2x + 5y + 3z = 2$$

$$-x - y = -1$$

We saw that the reduced row-echelon form of the augmented matrix here is

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The free variable is x_3, and there are two pivots. The solution set can be written as

$$(1+s, -s, s) \text{ where } s \text{ is any real number}$$

We can also write this as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The solution set of a non-homogeneous system is a **particular solution** (any single solution to the non-homogeneous equation) plus all solutions to the homogeneous system. In the example above, the vector $(1, 0, 0)$ is a particular solution, and the set

$$s \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \text{Span}\{(1, -1, 1)\}$$

is the solution set of the homogeneous equation (the system with the numbers on the other side of the equal sign all replaced by zeros). In particular, the homogeneous system for this problem is

$$x + 2y + z = 0$$

$$2x + 5y + 3z = 0$$

$$-x - y = 0$$

If you do enough of these particular solution plus homogeneous solution decompositions, you will notice that the solution to the homogeneous system is always a span. **This reflects the fact that a linear combination of any two solutions to the homogeneous system is also a solution of the homogeneous system. You show that this is true in Additional Problem 1 on Problem Set 1.**

Problem 2

Solve the following systems of equations.

$$x + y = 2$$

$$2x - y = 1$$

$$x - 2y = -1$$

(In particular, this example shows that if the reduced row-echelon form has a row of zeros, it is not necessarily true that the system has infinitely many solutions.)

$$x + 4y = -2$$
$$-2x - y = -3$$
$$3x + 2y = 4$$

$$x + 3y + z = 2$$
$$2x + 3y - 3z = 1$$

Problem 3

For what values of h is the following system consistent?

$$x + 2y + hz = 3$$
$$2x + 3y - z = 1$$
$$4x + 7y + 3z = 4$$

Problem 4

Show that an underdetermined homogeneous system (a system where there are more variables than equations) always has infinitely many solutions.

Problem 5

Find an example of an inconsistent homogeneous system of equations or show that one cannot exist (in which case every homogeneous system of equations is consistent).

Problem 6

Show by example that an overdetermined homogeneous system (a system where there are more equations than variables) can have either a unique solution, or infinitely many solutions.

Problem 7

Find conditions on a, b, and c such that (a, b, c) is a linear combination of the vectors $(1, 1, 0)$, $(-1, 2, 1)$, and $(1, 4, 1)$.